APPENDIX C. ONLINE APPENDIX

Appendix C.1. *Equivalence Ramsey and Constrained Social Planner*

The optimization problem of a Ramsey planner who has the tax instruments $(\tau_x^{i,\omega})$ and τ_k^i on date 0 security purchases and capital investment and agent specific transfers T^i is equivalent to the problem of a constrained social planner who directly chooses the economy's date 0 allocations $(C_0^i, K_1^i, X_1^{i,\omega})$, or equivalently $(C_0^i, K_1^b, N^{i,\omega})$ for $i \in I$ and $\omega \in \Omega$.

The budget constraint that incorporates taxes and transfers faced by agent i at date 0 is formally given by

$$c_0^i + h^i\left(k_1^i\right) + \tau_k^i k_1^i + \mathbb{E}_0\left[\left(m_1^\omega + \tau_x^{i,\omega}\right)x_1^{i,\omega}\right] = e_0^i + T_0^i$$

Formally, the optimality conditions of agents in a decentralized equilibrium with tax instruments $(\tau_x^{i,\omega})$ and (τ_k^i) are

$$\left(m_1^\omega + \tau_x^{i,\omega}\right)\lambda_0^i = \beta\lambda_1^{i,\omega} + \kappa_1^i \Phi_{1x^\omega}^i \quad \forall i,\omega$$

$$\left(h^{i\prime}\left(k_1^i\right) + \tau_k^i\right)\lambda_0^i = \beta\mathbb{E}_0\left[\lambda_1^{i,\omega}\left(F_1^{i,\omega\prime}\left(k_1^{i,\omega}\right) + q^\omega\right)\right] + \kappa_1^i\Phi_{1k}^i, \quad \forall i$$

Any date 0 allocation $(C_0^i, K_1^b, X_1^{i,\omega})$ chosen by a social planner can be replicated by a Ramsey planner who sets the tax instruments

$$\tau_x^{i,\omega} = \frac{\beta\lambda_1^{i,\omega}}{\lambda_0^i} + \frac{\kappa_1^i}{\lambda_0^i}\Phi_{1x}^{i,\omega} - m_1^\omega$$

$$\tau_k^i = \mathbb{E}_0\left[\frac{\beta\lambda_1^{i,\omega}}{\lambda_0^i}\left(F_1^{i,\omega\prime}\left(k_1^{i,\omega}\right) + q^\omega\right)\right] + \frac{\kappa_1^i}{\lambda_0^i}\Phi_{1k}^i - h^{i\prime}\left(k_1^i\right)$$

and who imposes transfers so that the date 0 budget constraints of individual agents are satisfied, where the state price densities equal the increase in the social planner's shadow prices on the resource constraint, $m_1^\omega = \nu_1^\omega/\nu_0$.[28]

Conversely, any set of tax instruments and transfers will result in a date 0 allocation $(c_0^i, k_1^i, x_1^{i,\omega})$ for $i \in I$ and $\omega \in \Omega$ that satisfies the economy's resource constraints. A constrained social planner can therefore replicate the allocation by setting her date 0 allocation $(C_0^i, K_1^i, X_1^{i,\omega})$ equal to that chosen by decentralized agents under the planner's optimal tax instruments.

Appendix C.2. *Welfare Weights and Pareto Improvements*

Proposition 1 characterizes constrained efficient allocations for given Pareto weights $\left(\theta^b, \theta^\ell\right)$ to describe the entire Pareto frontier of the economy, but only a subset of these allocations represent a Pareto improvement over laissez-faire.

In particular, imposing the optimal tax rates (27) and (28) without using transfers across borrowers and lenders, i.e. rebating all tax revenue to the set of agents from whom it is obtained, does not guarantee a Pareto improvement.[29] For an example, consider an economy in which the set of taxes $(\tau_x^{i,\omega}, \tau_k^i)$ implements a Pareto efficient allocation. Now assume that we move from an inefficient set of taxes where one tax rate $\hat{\tau}_x^{i,\omega_0}$ is marginally below its efficient level τ_x^{i,ω_0} to the described efficient set of taxes. Since we were close to efficiency, the overall welfare improvement is second-order, but the change in tax rates leads to a first-order change in equilibrium prices and a first-order redistribution between sectors. One sector will experience a first-order welfare gain, the other a first-order loss, implying that the policy change does not generate a Pareto improvement, even though it moves the economy from an inefficient allocation to a constrained Pareto efficient allocation.

To achieve a Pareto improvement, a planner has to pick relative welfare weights θ^ℓ/θ^b that correspond to the region of Pareto improvements, as illustrated by the section on the Pareto frontier in between the two circles in Figure 9. Formally, this region is characterized as follows. Let us normalize $\theta^b = 1$ and capture by the fraction $\theta = \theta^\ell/\theta^b$ how we vary the relative welfare weights on the two sectors. Let the planner solve

$$\max U^b \qquad \text{s.t.} \qquad U^\ell \geq U^{\ell,DE}$$

subject to the constraints of problem (24) and assign Lagrange multiplier $\underline{\theta}$ to the constraint $U^\ell \geq U^{\ell,DE}$ where $U^{i,DE}$ is the welfare of sector i agents in the decentralized equilibrium allocation. The solution to this optimization problem

28. See Application 3 and **?** for more detailed discussions regarding the role of transfers.
29. Rebating all tax revenue to the set of agents from whom it is obtained replicates the same allocation that would be obtained if the planner used quantity restrictions on date 0 allocations – therefore our statement on Pareto efficiency not guaranteeing a Pareto improvement extends to quantity regulations.

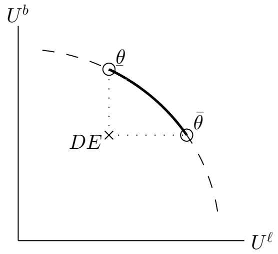

FIGURE 9
Pareto frontier

defines the minimum relative welfare weight on lenders $\underline{\theta}$ such that lenders are not made worse off by the planner's intervention.

Similarly, let the planner solve

$$\max U^\ell \qquad \text{s.t.} \qquad U^b \geq U^{b,DE}$$

subject to the constraints of problem (24) and assign Lagrange multiplier $1/\bar{\theta}$ to the constraint $U^b \geq U^{b,DE}$. This optimization problem defines the maximum relative welfare weight on lenders $\bar{\theta}$ such that borrowers are not made worse off by the planner's intervention.

Then we find following result, stated as a corollary of Propositions 1 and 2.[30]

Corollary 6. (Pareto improvements) *If the decentralized equilibrium allocation is constrained inefficient, then $\underline{\theta} < \bar{\theta}$ and a planner can achieve a Pareto improvement by imposing any set of relative welfare weights $\theta = \theta^\ell/\theta^b \in [\underline{\theta}, \bar{\theta}]$. She can achieve a strict Pareto improvement for any θ in the interior of the interval. Conversely, if the decentralized equilibrium is constrained efficient, then $\theta = \underline{\theta} = \bar{\theta}$ corresponds to the planner's relative welfare weights for the decentralized equilibrium allocation.*

Appendix C.3. *Generalizations of Baseline Model*

This appendix generalizes our baseline model along several dimensions and shows that the optimal corrective tax formulas (27) and (28) continue to apply. First, we allow for a more general set of agents with general discount factors and state-contingent utility that can capture subjective probabilities. Second, we allow for a more general investment and production structure. Finally, we also allow for a more general specification of financial constraints that now apply to all agents.

Preferences/endowments. Assume that there is a set of agents is given by $I = \{1, \dots |I|\}$. Agents have heterogeneous preferences, given by

$$U^i = \mathbb{E}_0 \left[\sum_{t=0}^{2} \left(\beta^i\right)^t u_t^i \left(c_t^i; \omega\right) \right] \tag{C62}$$

which allows both for agent-specific discount factors β^i and arbitrary time separable utility functions that may or may not satisfy an Inada condition $\lim_{c\to 0} u_t^{i\prime}(c;\omega) = \infty$. Since $u_t^i(\cdot;\omega)$ depends on the state of nature, the setup is also able to capture heterogeneous beliefs, i.e. agents may value consumption in some states of nature more highly because they believe them to be more likely. We continue to denote agent i endowment at date t given state ω by $e_t^{i,\omega}$. As in the baseline model, consumption must be non-negative, so $c_t^i \geq 0$.

30. It is not generally possible to obtain a simple explicit expression for the transfers required to obtain a Pareto improvement. However, the planner can approximate the wealth transfer that occurred in response to an intervention using the marginal distributive effects $\mathcal{D}_{N^i}^{j,\omega}$ times the change in sector-wide net worth $\Delta N^{i,\omega}$, and use the $MRS^{j,\omega}$ to discount this to date 0. If, for example, a macroprudential intervention increases the net worth of sector i by $\Delta N^{i,\omega}$, this suggests that a date 0 transfer of $-\mathbb{E}_0[MRS^{j,\omega}\mathcal{D}_{N^i}^{j,\omega}\Delta N^{i,\omega}]$ would leave type j agents approximately indifferent. A similar expression can be obtained for changes in sector-wide capital K_1^i.

Technology. Assume that each agent i is initially endowed with k_0^i units of capital and chooses to create ι_0^i capital goods at date 0 at a cost $h_0^i\left(\iota_0^i, k_0^i\right)$ and invest/disinvest into capital ι_1^i at cost $h_1^i\left(\iota_1^i, k_1^i\right)$ at date 1, where the cost functions satisfy $h_t^i\left(0, k_t^i\right) = 0$ and are increasing and convex in ι_t^i and decreasing in k_t^i. This formulation nests fixed capital endowments if we assume that $h_t^i\left(\iota_t^i, k_t^i\right) = \infty$ for $\iota_t^i \neq 0$ for $t = 0, 1$. It also nests standard quadratic adjustment costs if we assume that $h_1^i(\iota_1^i, k_1^i) = \iota_1^i(1 + (\varphi/2) \cdot \iota_1^i/k_1^i)$, as well as a range of other models of endogenous investment with and without adjustment costs. We continue to assume that capital fully depreciates at the end of date 2.

Regarding production, we assume that type i agents have access to a production technology given by the function $F_t^{i,\omega}(k)$ that is increasing and weakly concave and depends on the agent type i, the date t, and the state of nature ω. As we discussed in the main text, it is common in the literature on fire sales to assume that the productivity of capital depends on who owns it and differs between different agents. This rules out that capital is owned by one agent but rented out and used in another agent's production function. A typical justification for this assumption is agency frictions, i.e. that efficient use of capital requires ownership to ensure proper incentives (see e.g. **?**).

The budget constraints of type i agents, including tax instruments and transfers, which generalize equations (2) to (4) in the text, are given by

$$c_0^i + h_0^i\left(\iota_0^i, k_0^i\right) + \left(q_0 + \tau_k^i\right)\left(k_1^i - k_0^i - \iota_0^i\right) \qquad \text{(C63)}$$

$$+\mathbb{E}_0\left[\left(m_1^\omega + \tau_x^{i,\omega}\right)x_1^{i,\omega}\right] = e_0^i + F_0^i\left(k_0^i\right) + T^i$$

$$c_1^{i,\omega} + h_1^i\left(\iota_1^{i,\omega}, k_1^i\right) + q_1^\omega\left(k_2^{i,\omega} - k_1^i - \iota_1^{i,\omega}\right) + m_2^\omega x_2^{i,\omega} = e_1^{i,\omega} + x_1^\omega + F_1^{i,\omega}\left(k_1^i\right),\ \forall\omega \qquad \text{(C64)}$$

$$c_2^{i,\omega} = e_2^{i,\omega} + x_2^\omega + F_2^{i,\omega}\left(k_2^{i,\omega}\right),\ \forall\omega \qquad \text{(C65)}$$

Financial constraints. We generalize financial constraints along three dimensions. First, we assume that $\Phi_2^{i,\omega}(\cdot)$ depends directly on the aggregate state variables N^ω and K_1, not only through the date 1 asset price. This more general formulation emphasizes the role played by the dependence of financial constraints on aggregate state variables and allows for example the bond price m_2^ω to enter the constraint. It can similarly capture more general moral hazard and incentive constraints. Secondly, we assume that $\Phi_1^{i,\omega}$ also depends on the entire vector of future aggregate state variables N^ω and K_1. This formulation allows us to capture date 0 financial constraints that depend on future date 1 prices, which can be written as a function of these aggregate state variables. Third, we assume that all agents, not only borrowers, are potentially subject to financial constraints. Formally, agent i faces vector-valued financial constraints

$$\Phi_1^i\left(x_1^i, k_1^i; N^\omega, K_1\right) \geq 0 \qquad \text{(C66)}$$

$$\Phi_2^{i,\omega}\left(x_2^i, k_2^i; N^\omega, K_1\right) \geq 0,\ \forall\omega \qquad \text{(C67)}$$

Equilibrium at dates 1 and 2. Agent i maximizes the generalized utility function (C62) subject to the set of budget constraints (C63), (C64), (C65) and the set of financial constraints (C66) and (C67).

As in the baseline model, the vectors $N^\omega = (N^{1,\omega}, ...N^{|I|,\omega})$ and $K_1 = (K_1^1, ...K_1^{|I|})$, representing aggregate net worth and capital holdings of type i agents, become aggregate state variables. We continue to index indirect utility by the state ω, to capture the direct effect of the state on investment opportunities.

The date 1 continuation utility $V^{i,\omega}\left(n^i, k_1^i; N^\omega, K_1\right)$ of type i agents is

$$V^{i,\omega}\left(n^{i,\omega}, k_1^{i,\omega}; N^\omega, K_1\right) = \max_{c_1^{i,\omega}\geq 0, c_2^{i,\omega}\geq 0, x_2^{i,\omega}, \iota_1^{i,\omega}, k_2^{i,\omega}} u_1^i\left(c_1^{i,\omega}; \omega\right) + \beta^i u_2^i\left(c_2^{i,\omega}; \omega\right)$$

s.t. (C64), (C65) and (C67)

Date 1 market prices are now functions $q_1^\omega(N^\omega, K_1)$ and $m_2^\omega(N^\omega, K_1)$ of the net worth and capital holding vectors of all sectors. Agent i date 1 optimality conditions for borrowing/saving and capital holdings can be expressed as

$$\lambda_1^{i,\omega} m_2^\omega = \beta^i \lambda_2^{i,\omega} + \kappa_2^{i,\omega}\Phi_{2x}^{i,\omega}$$

$$\lambda_1^{i,\omega} q_1^\omega = \beta^i \lambda_2^{i,\omega} F_2^{i,\omega\prime}\left(k_2^{i,\omega}\right) + \kappa_2^{i,\omega}\Phi_{2k}^{i,\omega}$$

The optimality condition for investment is given by $q_1^{\omega} = \partial h^i / \partial \iota_1^{i,\omega}$. Equations (14) and (15) remain valid in this more general environment, with the exception that the collateral effects now correspond to

$$\mathcal{C}_{N^j}^{i,\omega} := \frac{\partial \Phi_2^{i,\omega}}{\partial N^{j,\omega}}$$
$$\mathcal{C}_{K^j}^{i,\omega} := F_1^{i,\omega\prime}(\cdot)\,\mathcal{C}_{N^j}^{i,\omega} + \frac{\partial \Phi_2^{i,\omega}}{\partial K_1^j}$$

Date 0 decentralized equilibrium. At date 0, private agents of type i solve the maximization problem

$$\max_{c_0^i \geq 0, \iota_0^i, k_1^i, x_1^{i,\omega}} u_0^i\left(c_0^i\right) + \beta^i \mathbb{E}_0\left[V^{i,\omega}\left(x_1^{i,\omega} + e_1^{i,\omega} + F_1^{i,\omega}\left(k_1^i\right), k_1^i; N^{\omega}, K_1\right)\right] \quad \text{s.t.} \quad \text{(C63), (C66)}$$

At date 0, agent i optimality conditions can be expressed as

$$\left(m_1^{\omega} + \tau_x^{i,\omega}\right) u_0^{i\prime}\left(c_0^i\right) = \beta^i u_1^{i\prime}\left(c_1^{i,\omega};\omega\right) + \kappa_1^i \Phi_{1x}^{i,\omega},\ \forall \omega \tag{C68}$$
$$\left(q_0 + \tau_k^i\right) u_0^{i\prime}\left(c_0^i\right) = \mathbb{E}_0\left[\beta^i u_1^{i\prime}\left(c_1^{i,\omega};\omega\right)\left(F_1^{i,\omega\prime}\left(k_1^i\right) + q_1^{\omega} - \frac{\partial h_1^i}{\partial k_1^i}\right)\right] + \kappa_1^i \Phi_{1k}^i \tag{C69}$$

The optimality condition for investment is given by $q_0 + \tau_k^i = \partial h^i / \partial i_0^i$.

Date 0 constrained planner allocation. The Lagrangian corresponding to the problem solved by a constrained planner who leaves date 1 allocations to the market but determines date 0 allocations is given by

$$\mathcal{L} = \sum_i \theta^i \left\{u^i\left(C_0^i\right) + \eta_0^i C_0^i + \beta^i \mathbb{E}_0\left[V^{i,\omega}\left(N^{i,\omega}, K_1^i; N^{\omega}, K_1\right)\right] + \kappa_1^i \Phi_1^i\left(X_1^i, K_1^i; N^{\omega}, K_1\right)\right\}$$
$$- \nu_0 \sum_i \left[C_0^i + h_0^i\left(\iota_0^i, K_0^i\right) - e_0^i - F_0^i\left(K_0^i\right)\right] - \mu_0 \sum_i \left[K_1^i - K_0^i - \iota_0^i\right] - \sum_{\omega} \nu_1^{\omega} \sum_i X_1^{i,\omega}$$

We assign the new shadow price μ_0 to the new constraint on capital accumulation. The optimality conditions of the planner problem are given by

$$\frac{d\mathcal{L}}{dC_0^i} = \theta^i\left[u_0^{i\prime}\left(C_0^i\right) + \eta_0^i\right] - \nu_0 = 0,\ \forall i$$
$$\frac{d\mathcal{L}}{dX_1^{i,\omega}} = -\nu_1^{\omega} + \theta^i \beta^i V_n^{i,\omega} + \theta^i \kappa_1^i \Phi_{1x}^{i,\omega} + \sum_j \theta^j \beta^j V_{N^i}^{j,\omega} + \sum_j \theta^j \kappa_1^j \frac{\partial \Phi_1^j}{\partial N^{i,\omega}} = 0,\ \forall i,\omega$$
$$\frac{d\mathcal{L}}{dK_1^i} = -\mu_0 + \theta^i \beta^i \mathbb{E}_0\left[V_k^{i,\omega}\right] + \theta^i \kappa_1^i \Phi_{1k}^i + \beta \sum_j \theta^j \mathbb{E}_0\left[V_{N^i}^{j,\omega} F_1^{i,\omega\prime}\left(K_1^i\right) + V_{K^i}^{j,\omega}\right]$$
$$+ \sum_j \theta^j \kappa_1^j \left(\sum_{\omega} \frac{\partial \Phi_1^j}{\partial N^{i,\omega}} F_1^{i,\omega\prime}(\cdot) + \frac{\partial \Phi_1^j}{\partial K_1^i}\right) = 0,\ \forall i$$

Where also $d\mathcal{L}/d\iota_0^i = -\nu_0 \partial h_0^i / \partial \iota_0^i + \mu_0$, and we denote $V_n^{i,\omega} = u_1^{i\prime}(C_1^{i,\omega})$ and $V_k^{i,\omega} = F_1^{i,\omega\prime}(k_1^i) + q^{\omega} - \partial h_1^i / \partial k_1^i$. Rearranging the optimality conditions and the date 1 envelope conditions, we find the analogous equations to (25) and (26) characterizing constrained efficiency in this more general version

$$\frac{\nu_1^{\omega}}{\nu_0} \lambda_0^i = \beta^i \lambda_1^{i,\omega} + \kappa_1^i \Phi_{1x^{\omega}}^i + \sum_{j \in I} \frac{\theta^j}{\theta^i} \beta^j V_{N^i}^{j,\omega},\ \forall i,\omega \tag{C70}$$
$$\frac{\partial h_0^i}{\partial i_0^i} \lambda_0^i = \mathbb{E}_0\left[\beta^i \lambda_1^{i,\omega}\left(F_1^{i,\omega\prime}\left(k_1^i\right) + q^{\omega} - \frac{\partial h_1^i}{\partial k_1^i}\right)\right] \tag{C71}$$
$$+ \kappa_1^i \Phi_{1k}^i + \sum_j \frac{\theta^j}{\theta^i} \beta^j \mathbb{E}_0\left[V_{N^i}^{j,\omega} F_1^{i,\omega\prime}\left(K_1^i\right) + V_{K^i}^{j,\omega}\right]$$
$$+ \sum_j \frac{\theta^j}{\theta^i} \kappa_1^j \left(\sum_{\omega} \frac{\partial \Phi_1^j}{\partial N^{i,\omega}} F_1^{i,\omega\prime}(\cdot) + \frac{\partial \Phi_1^j}{\partial K_1^i}\right)$$

The only difference with the baseline model is a new collateral externality term, capturing the direct effect of changes in aggregate state variables on the date 1 financial constraints – this is a straightforward generalization of the results in the text. As in the baseline model, the planner can set optimal corrective taxes to ensure that the private optimality conditions (C68) and (C69) replicate the planner's optimality conditions (C70) and (C71). Propositions 1 and 2 and all their implications, including the characterization of optimal corrective taxes in (27) and (28) and the corollaries, remain valid in this more general environment after accounting for the new collateral externality term. In the case with I agents, $I - 1$ differences in MRS are needed to express the optimal tax wedges as in equations (29) and (30).