

# Pecuniary Externalities in Economies with Financial Frictions

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This paper characterizes the efficiency properties of competitive economies with financial constraints, in which phenomena such as fire sales and financial amplification may arise. We show that financial constraints lead to two distinct types of pecuniary externalities: distributive externalities that arise from incomplete insurance markets and collateral externalities that arise from price-dependent financial constraints. For both types of externalities, we identify three sufficient statistics that determine optimal taxes on financing and investment decisions to implement constrained efficient allocations. We also show that fire sales and financial amplification are neither necessary nor sufficient to generate inefficient pecuniary externalities. We demonstrate how to employ our framework in a number of applications. Whereas collateral externalities generally lead to over-borrowing, the distortions from distributive externalities may easily flip sign, leading to either under- or over-borrowing. Both types of externalities may lead to under- or over-investment.

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## 1. INTRODUCTION

Modern economies have experienced recurrent financial crises involving sharp drops in asset prices and amplification effects. Policy discussions in the aftermath of the 2008/09 Global Financial Crisis have understandably focused on the possibility that such “fire sales” may lead to inefficient externalities that call for regulatory intervention – as exemplified by the speech by Stein (2013). Understanding whether financial amplification and fire sales, i.e. asset sales at dislocated prices by financially constrained agents, provide a rationale for policy intervention is thus crucial to redesigning our financial regulatory framework.

In the existing literature, the seminal papers of Gromb and Vayanos (2002) and Lorenzoni (2008) describe how asset sales by financially constrained agents can generate pecuniary externalities that lead to constrained inefficient allocations.<sup>1</sup> Some policymakers and commentators have interpreted this as implying that sharp changes in prices always involve inefficient externalities. However, the efficiency properties of economies with financially constrained agents are less obvious than commonly understood, and a general description of the resulting externalities has been missing.

1. Whenever some agents are financially constrained, the market outcome is clearly not first-best: removing the frictions that underlie the financial constraints increases efficiency. However, in practice, policymakers frequently must take such frictions as given, which leads to the question of whether decentralized equilibrium allocations are *constrained efficient*. In other words, can a policymaker subject to the same constraints as private agents improve on the market outcome?

This paper seeks to fill this gap by developing a general framework to characterize the pecuniary externalities that arise in environments with financially constrained agents. Our first main result characterizes constrained efficient allocations and optimal corrective policies with borrowers who are subject to financial constraints. We describe the optimal corrective policies for financing and investment decisions as a function of sufficient statistics that are invariant to the precise nature of the underlying financial frictions, e.g., uncollateralized bonds, limited commitment, market segmentation, etc.<sup>2</sup>

We show that two distinct types of pecuniary externalities arise in such environments. We refer to the first type as *distributive externalities* to highlight that these externalities are zero-sum across agents at a given date/state. Distributive externalities arise when marginal rates of substitution (MRS) between dates/states differ across agents, and a planner can improve on the allocation by affecting the relative prices at which agents trade. Potential reasons why the MRS are not equalized include, for instance, that the set of traded assets does not span all possible states of nature, or binding collateral constraints. Intuitively, when MRS are not equal, a planner can modify allocations to induce price changes that improve the terms of the transactions of those agents with relatively higher marginal utility in a given date/state. For example, a planner may internalize that reducing fire sales raises the price received by the sellers, who may greatly value having resources in those states as reflected by a high MRS.

We refer to the second type as *collateral externalities*. Collateral externalities arise when financial constraints depend on the market value of capital assets that serve as collateral. They are part of a broader class of externalities that arise when financial constraints depend on aggregate state variables, for example via market prices, which we analyze in the appendix. Intuitively, when agents are subject to a binding constraint that depends on aggregate variables, a planner internalizes that she can modify allocations to relax financial constraints. For example, the planner may reduce fire sales to raise the value of capital assets that serve as collateral, which increases the borrowing capacity of constrained agents.

The existing literature has found it remarkably difficult to provide general results on the direction of inefficiency – except in tightly-defined special cases. Our second main result explains why and delineates under what conditions the pecuniary externalities can be signed unambiguously and when they can go in either direction. The sign and magnitude of distributive externalities are determined by the product of three sufficient statistics: the difference in MRS of agents, the net trading positions (net buying or net selling) of capital and financial assets, and the sensitivity of equilibrium prices to changes in sector-wide state variables. The first two of the three sufficient statistics for distributive externalities can go in either direction. Depending on parameters, it is plausible to find economies in which differences in MRS and net trading positions take positive or negative values. Furthermore, if risk markets are complete, MRS are equated and distributive externalities are zero. In short, “anything goes,” and distributive externalities cannot be signed in general.

The sign and magnitude of collateral externalities is also determined by the product of three sufficient statistics: the shadow value on the binding financial constraint, the sensitivity of the financial constraint to the asset price, and the sensitivity of the equilibrium asset price to changes in sector-wide state variables. The first two of the three sufficient statistics for collateral externalities are always positive. Under natural conditions, asset prices are increasing in net worth for each sector, pinning down the sign of the third sufficient statistic. This allows us to show that collateral externalities generally entail over-borrowing, but they may lead to either over- or under-investment. Importantly, our characterization of both distributive and collateral

2. We adopt the concept of *sufficient statistics* to refer to high-level variables, as opposed to primitives, that determine, within the environment we study, the presence of pecuniary externalities and the nature of the optimal corrective policy. In our applications, we link the sufficient statistics that we identify to primitives of the model.

externalities holds in a broad class of environments and is invariant to the precise nature of the underlying financial frictions.

We present two results on the implementation of corrective policies. First, we show that the optimal corrective policy for an arbitrary financial security can be designed using an externality pricing kernel. This result provides a simple expression to guide financial regulators on the optimal magnitude of regulatory interventions. Secondly, we show that there exists a relation between distortions in investment in productive assets and distortions in financial market allocations. Intuitively, because investing in productive assets and buying financial assets are both mechanisms for shifting resources across time, optimal policies must intervene in both margins in a consistent way.

Next, we discuss the relationship with two positive phenomena that are distinct from pecuniary externalities but frequently appear in the same context: fire sales and financial amplification. For the purposes of our framework, we define fire sales as instances when financially constrained agents sell capital assets at a price that discounts the future returns that they could earn at a higher rate than the market discount rate. We define financial amplification as a situation when a marginal increase in the net worth of a sector, as measured by the consumption goods at its disposal, leads to general equilibrium effects that improve the sector's terms of trade or relax binding financial constraints on the sector. We show that both fire sales and financial amplification effects are conceptually distinct phenomena from inefficient pecuniary externalities. Formally, both phenomena are neither necessary nor sufficient for constrained inefficiency. They are not necessary because inefficiency may arise without asset sales and may involve pecuniary externalities that mitigate shocks rather than amplifying them. They are not sufficient because equilibrium is constrained efficient when there are fire sales and amplification effects that only involve distributive externalities and insurance markets are complete, or when agents are in a corner solution. This result implies that policymakers have to be careful when arguing that fire sales and financial amplification effects justify policy intervention.

Finally, we show that the externalities discussed above can be tackled by a variety of taxes or subsidies on borrowers and lenders. In particular, the planner faces three degrees of freedom in the choice of a constrained optimal tax system. This flexibility allows a planner to restore constrained efficiency without intervening in each individual decision made by each agent. For example, we show that it is sufficient to intervene in the financial decisions of borrowers only, or that we can often combine taxes on borrowing and on investment into a single tax. Furthermore, when these degrees of freedom imply that the optimal tax on a decision margin can be set to zero, that decision can be interpreted as constrained efficient.

Subsequently, we study four applications of our general framework that illustrate the use of our sufficient statistics and how they can be traced back to the primitives of the economy. In doing so, we also provide specific examples of how some of our sufficient statistics may flip sign when the primitives of the model cross a defined threshold, corroborating the "anything goes" result of our general framework.

Our first application illustrates the possibility of constrained efficient financial amplification and fire sales. In an environment in which the financial constraint does not depend on prices and risk markets are complete, we show that fire sales and financial amplification effects of arbitrary magnitude are compatible with constrained efficiency. The reason is that the complete risk markets allow agents to equate their MRS so distributive effects do not lead to inefficiency. Our second and third applications consider environments in which there are distributive externalities that flip sign when certain primitives of the economy cross well-defined thresholds. In the second application, borrowers turn from net buyers into net sellers of capital when a productivity parameter crosses a certain threshold. In the third application, the

difference in the MRS of borrowers and lenders switches sign as borrowers hit the upper versus the lower limit for trade in the constrained financial market when their endowment crosses two well-defined thresholds. When the sufficient statistics flip sign, the direction of inefficiency of financing and investment decisions switches sign as well. Our fourth application provides an example of a price-dependent collateral constraint in which collateral externalities cause over-borrowing and either over- or under-investment. At last, we map our applications to real-world situations.

Before concluding, we use the general framework developed in the paper to place in context several results highlighted by previous literature. In particular, we classify papers according to whether they focus on distributive or collateral externalities or both.

*Outline.* Section 2 describes the baseline model environment, characterizes the first best and solves for the decentralized equilibrium. We study the constrained efficiency properties of the equilibrium and present several corollaries in Section 3. In Section 4, we illustrate our findings in a number of specific applications. Section 5 relates our results to previous work, and Section 6 concludes. All proofs and derivations as well as several extensions are in the appendix.

## 2. BASELINE MODEL

Our baseline model describes fire sales in an economy with two types of agents that we call borrowers and lenders. Borrowers are potentially more productive than lenders at using capital but are subject to financial constraints that may lead to fire sales. The model environment can be viewed as a simplified three-date version of Kiyotaki and Moore (1997) with alternative preferences, technology, and financial market structure.<sup>3</sup>

### 2.1. Environment

Time is discrete and there are three dates  $t = 0, 1, 2$ . There is a unit measure of borrowers and a unit measure of lenders, respectively denoted by  $i \in I = \{b, \ell\}$ . There are two types of goods, a homogeneous consumption good, which serves as numeraire, and a capital good. We denote by  $\omega \in \Omega$  the state of nature realized at date 1, where  $\Omega$  is the set of possible states.

*Preferences/endowments.* Each agent  $i$  values consumption  $c_t^i \geq 0$  according to a time separable utility function

$$U^i = \mathbb{E}_0 \left[ \sum_{t=0}^2 \beta^t u^i(c_t^i) \right] \quad (1)$$

where the flow utility function  $u^i(c)$  is strictly increasing and weakly concave. We denote by  $e_t^{i,\omega} \geq 0$  the endowment of consumption good that agent  $i$  receives at date  $t$  given a state  $\omega$ .

*Technology.* At date 0, agents can invest  $h^i(k_1^i)$  units of consumption good to produce  $k_1^i$  units of date 1 capital goods, where the functions  $h^i(k)$  are increasing and convex and satisfy  $h^i(0) = 0$ . The economy's total capital stock remains constant at  $k_1^b + k_1^\ell$  after the initial investment. We denote by  $k_2^{i,\omega}$  the amount of capital that agent  $i$  carries from date 1 to 2. Capital fully depreciates after date 2.

3. For expositional simplicity, our baseline model only features two agents and a specific production structure. We extend our main results to multiple agents with more general state-dependent utilities and a more general investment and production structure in the online appendix.

At dates 1 and 2, agent  $i$  employs capital to produce  $F_t^{i,\omega}(k)$  units of the consumption good, where the production function is increasing and weakly concave and satisfies  $F_t^{i,\omega}(0) = 0$ . As is common in the literature on fire sales, we assume that the productivity of capital depends on who owns it (see e.g. Shleifer and Vishny, 1992). We will typically assume that borrowers have a superior use for capital goods than lenders in our applications.

*Market structure.* At date 0, agents trade one-period securities contingent on every state of nature  $\omega \in \Omega$ . We denote by  $x_1^{i,\omega}$  the date 0 purchases of state  $\omega$  contingent securities by agent  $i$  and by  $m_1^\omega$  the date 0 state price density associated with such securities. If  $x_1^{i,\omega} < 0$ , agent  $i$  borrows against state  $\omega$ . If  $x_1^{i,\omega} > 0$ , agent  $i$  saves towards state  $\omega$ . The total amount spent by agent  $i$  at date 0 on state-contingent securities is  $\mathbb{E}_0[m_1^\omega x_1^{i,\omega}]$ . Because there is no further uncertainty at date 2, we denote by  $x_2^{i,\omega}$  the date 1 holdings of uncontingent one-period bonds in state  $\omega$ , which trade at a price  $m_2^\omega$ . There is also a market to trade capital at a price  $q^\omega$  at date 1 after production has taken place. There is no role for trading capital at date 2 because it fully depreciates.

The budget constraints capture that consumption, capital investment, and net purchases of capital and securities need to be covered by endowment income, security payoffs, and production income for each agent  $i$  in every state  $\omega \in \Omega$

$$c_0^i + h^i(k_1^i) + \mathbb{E}_0[m_1^\omega x_1^{i,\omega}] = e_0^i \quad (2)$$

$$c_1^{i,\omega} + q^\omega \Delta k_2^{i,\omega} + m_2^\omega x_2^{i,\omega} = e_1^{i,\omega} + x_1^{i,\omega} + F_1^{i,\omega}(k_1^i), \quad \forall \omega \quad (3)$$

$$c_2^{i,\omega} = e_2^{i,\omega} + x_2^{i,\omega} + F_2^{i,\omega}(k_2^{i,\omega}), \quad \forall \omega \quad (4)$$

where  $\Delta k_2^{i,\omega} := k_2^{i,\omega} - k_1^i$ . All choice variables at dates 1 and 2 are contingent on the state of nature  $\omega$ , which is realized at date 1.

*Financial constraints.* The final ingredient of our model is a set of financial market imperfections that constrain borrowers' choices. We introduce these through two vector-valued functions  $\Phi_1^b(\cdot)$  and  $\Phi_2^{b,\omega}(\cdot)$ .

At date 0, borrowers' security holdings  $x_1^b = (x_1^{b,\omega})_{\omega \in \Omega}$  are subject to a constraint of the form

$$\Phi_1^b(x_1^b, k_1^b) \geq 0 \quad (5)$$

which defines a convex set. At date 1, borrowers' security holdings  $x_2^{b,\omega}$  are subject to a possibly state-dependent constraint that is also a function of the asset price  $q^\omega$

$$\Phi_2^{b,\omega}(x_2^{b,\omega}, k_2^{b,\omega}; q^\omega) \geq 0, \quad \forall \omega \quad (6)$$

which defines a convex set and satisfies  $\Phi_{2q}^{b,\omega} := \partial \Phi_2^{b,\omega} / \partial q^\omega \geq 0$ . This sign restriction implies that a higher price of the capital good weakly relaxes the financial constraint.<sup>4</sup> For instance, if borrowers have to collateralize their borrowing with a fraction  $\phi^\omega \in [0, 1]$  of their asset

4. For expositional simplicity, the financial constraint at date 0 does not depend on prices or other aggregate variables in our baseline model. We show in the online appendix that it is straightforward to extend our results to that case. We also show that it is straightforward to allow for constraints that depend on future aggregate state variables, which is appropriate when financial constraints depend directly on future asset prices.

holdings,  $\Phi_2^{b,\omega}(\cdot) := x_2^{b,\omega} + \phi^\omega q^\omega k_2^{b,\omega} \geq 0$ . For symmetry of notation, we define  $\Phi_1^\ell(\cdot) = \Phi_2^{\ell,\omega}(\cdot) := 0$  so the constraints are always trivially satisfied for lenders.<sup>5</sup>

*Interpretation of financial constraints.* This general specification allows us to consider a wide range of financial constraints.<sup>6</sup> Focusing on the date 0 constraints, one extreme, captured by the specification  $\Phi_1^b(x_1^b, k_1^b) := 0$ , is that agents face no constraints at date 0 and can trade in a complete market, since constraint (5) becomes redundant under this specification. This can be interpreted as well-functioning risk markets. The opposite extreme, captured by the specification  $\Phi_1^b(x_1^b, k_1^b) := (x_1^{b,\omega})_{\omega \in \Omega}$  and the vector constraint  $\Phi_1^b(\cdot) = 0$  with equality, implies that no financial trade is possible and borrowers have to satisfy  $x_1^{b,\omega} = 0, \forall \omega$ . This can be interpreted as a severe disruption of financial markets. Clearly, a planner who is subject to the same constraint cannot alter the financing decisions of agents who face this constraint.

The most interesting cases are in between, when borrowers face some market incompleteness but still have some meaningful financing and investment decisions. Our framework can flexibly accommodate intermediate degrees of financial market imperfections, including different types of market incompleteness. For example, if we specify  $\Phi_1^b(x_1^b, k_1^b) := (x_1^{b,\omega} - x_1^{b,\omega_0})_{\omega \in \Omega \setminus \omega_0}$ , then the vector constraint  $\Phi_1^b(\cdot) = 0$  describes that borrowers can only trade bonds at date 0 – all state-contingent payments have to be identical,  $x_1^{b,\omega} = x_1^{b,\omega_0}$ . Alternatively, for the specification  $\Phi_1^b(x_1^b, k_1^b) := (x_1^{b,\omega} - \bar{x})_{\omega \in \Omega}$ , where  $\bar{x} < 0$ , the vector inequality constraint  $\Phi_1^b(\cdot) \geq 0$  captures a form of limited commitment on date 1 repayments, such that borrowers cannot promise to repay more than  $\bar{x}$ .

*Interpretation of environment.* Our baseline model captures a number of different situations in which financial constraints matter and fire sales may occur. We provide four natural interpretations. First, we can think of borrowers as entrepreneurs/firms who have a more productive use of capital goods than other agents in the economy. When financial constraints force them to sell, capital is diverted to a less efficient technology, leading to price declines. Second, borrowers can be interpreted as an amalgamate of financial intermediaries and firms that channel funds from savers/lenders into productive capital investment. If financial constraints force the intermediaries to reduce credit to the real sector, the firms are less able to externally finance their investments, leading to inefficient sales of capital. Third, we can also interpret borrowers as homeowners who hold mortgages. The transfer of houses from borrowers to lenders in case of foreclosure can accelerate house depreciation, causing declines in house prices. Finally, more broadly, when agents have heterogeneous preferences, we can interpret borrowers as financial specialists who place a higher value on risky assets than their lenders because they have a better capacity to bear risk, but who may be forced to unwind their positions at unusually low prices after a common negative shock.

5. We extend our results to the case in which both borrowers and lenders face financial constraints in the online appendix, and our propositions and corollaries continue to hold. The results of the baseline model can be interpreted as describing lenders that are subject to financial constraints but have sufficiently large endowments so that the constraints are not binding for them.

6. We have directly formulated financial constraints in the context of single-period claims. These types of constraints arise endogenously in some environments – see, for instance, the model of limited commitment without exclusion of Rampini and Viswanathan (2010) – however, multi-period constraints may arise in more general environments. The results of the paper can be adapted to that context. In particular, the sufficient statistics identified in this paper would remain valid in the more general case.

## 2.2. First Best

A real allocation is a bundle of consumption vectors  $(c_0^i, c_1^{i,\omega}, c_2^{i,\omega})$  and capital holdings  $(k_1^i, k_2^{i,\omega})$  for all  $\omega \in \Omega$  and  $i \in I$ . A real allocation is first-best if it maximizes the weighted sum of welfare  $\sum_i \theta^i U^i$  for some welfare weights  $\theta^b, \theta^\ell > 0$  subject to the resource constraints

$$\sum_i \left[ c_0^i + h^i(k_1^i) \right] \leq \sum_i e_0^i \quad (7)$$

$$\sum_i c_t^{i,\omega} \leq \sum_i \left[ e_t^i + F_t^{i,\omega}(k_t^{i,\omega}) \right], \text{ for } t = 1, 2 \text{ and } \forall \omega \quad (8)$$

$$\sum_i k_2^{i,\omega} \leq \sum_i k_1^i, \forall \omega \quad (9)$$

It is easy to see that a real allocation is first-best if it satisfies the resource constraints, if the marginal rates of substitution (MRS) between the two sets of agents are equated across time and states,

$$\frac{u^{b'}(c_0^b)}{u^{\ell'}(c_0^\ell)} = \frac{u^{b'}(c_1^{b,\omega})}{u^{\ell'}(c_1^{\ell,\omega})} = \frac{u^{b'}(c_2^{b,\omega})}{u^{\ell'}(c_2^{\ell,\omega})}, \forall \omega$$

if the marginal cost of capital investment equals its discounted expected benefit,  $u^{i'}(c_0^i)h^{i'}(k_1^i) = \mathbb{E}_0[\beta u^{i'}(c_1^{i,\omega})F_1^{i,\omega'}(k_1^i) + \beta^2 u^{i'}(c_2^{i,\omega})F_2^{i,\omega'}(k_2^{i,\omega})]$ ,  $\forall i$ , and if the marginal products of capital are equated at date 2,  $F_2^{i,\omega'}(k_2^{i,\omega}) = F_2^{j,\omega'}(k_2^{j,\omega})$ ,  $\forall i, j$ .

## 2.3. Decentralized Equilibrium

A decentralized equilibrium consists of a real allocation  $(c_0^i, c_1^{i,\omega}, c_2^{i,\omega}, k_1^i, k_2^{i,\omega})$ , a security allocation  $(x_1^{i,\omega}, x_2^{i,\omega})$ , together with a set of prices  $(m_1^\omega, m_2^\omega, q^\omega)$  such that both sets of agents solve their optimization problem and markets clear, i.e. equations (7), (8), and (9) hold, and  $\sum_i x_t^{i,\omega} = 0$  holds at dates 1 and 2,  $\forall \omega$ . For the rest of the paper, we proceed under the presumption that there exists a unique equilibrium.<sup>7</sup> When financial constraints never bind, the real allocation of the decentralized equilibrium of our economy is first-best.

We solve for the decentralized equilibrium via backward induction, paying particular attention to date 1, which is when pecuniary externalities materialize.

*Date 2 equilibrium.* Equilibrium at date 2 is simple. After production has taken place, agents settle their security positions and consume their holdings of consumption goods. Capital is worthless after date 2, since there is no further production in the economy.

*Date 1 equilibrium.* The state of the economy at date 1 is fully described by two sets of state variables: the net worth

$$n^{i,\omega} := e_1^{i,\omega} + x_1^{i,\omega} + F_1^{i,\omega}(k_1^i) \quad (10)$$

in terms of consumption goods (not including capital holdings), and the capital holdings  $k_1^i$  of both groups of agents. The agents' net worth fully captures the impact of uncertainty on the economy. Note that  $n^{i,\omega}$  may be negative if  $x_1^{i,\omega}$  or  $F_1^{i,\omega'}(k_1^i)$  is sufficiently negative – in that

7. At this level of generality, equilibrium existence and uniqueness are not guaranteed. Under regularity conditions, the generic existence results discussed, for instance, in Magill and Quinzii (2002) apply to our environment. We carefully establish the regularity properties of the model in each of our applications. We also provide examples of non-uniqueness in the appendix.

case, the agents need to borrow and/or fire-sell at date 1 to service existing debt or maintain their capital holdings.

It is useful to distinguish between individual state variables  $(n^{b,\omega}, n^{\ell,\omega}, k_1^b, k_1^\ell)$  and sector-wide aggregate state variables  $= (N^{b,\omega}, N^{\ell,\omega}, K_1^b, K_1^\ell)$ , which we denote by capitalized letters. In a symmetric equilibrium, it is always the case that  $n^{i,\omega} = N^{i,\omega}$  and  $k_1^i = K_1^i, \forall i, \omega$ . However, the distinction matters because individual agents take sector-wide variables as given whereas they internalize that they can affect their own state variables through their date 0 actions. Sector-wide variables enter the welfare function of individual agents since they affect the prices of capital and financial securities. This plays a crucial role in our analysis of externalities below. In the following, we collect the sector-wide net worth and capital holdings of borrowers and lenders at date 1 in the two vectors  $N^\omega = (N^{b,\omega}, N^{\ell,\omega})$  and  $K_1 = (K_1^b, K_1^\ell)$ .

We describe the date 1 optimization problem of an individual agent  $i$  as a function of both sets of state variables

$$V^{i,\omega}(n^{i,\omega}, k_1^{i,\omega}; N^\omega, K_1) = \max_{c_1^{i,\omega} \geq 0, c_2^{i,\omega} \geq 0, k_2^{i,\omega}, x_2^{i,\omega}} u^i(c_1^{i,\omega}) + \beta u^i(c_2^{i,\omega}) \quad \text{s.t. (3), (4) and (6)} \quad (11)$$

where we denote by  $\lambda_i^{i,\omega}$  the multipliers on the budget constraints (3) and (4), by  $\kappa_2^{b,\omega}$  the multiplier on borrowers' financial constraint (6), and by  $\eta_i^{i,\omega}$  the multipliers on the non-negativity of consumption constraints.<sup>8</sup> We define  $\kappa_2^{\ell,\omega} := 0$  to keep our notation symmetric. Since there is no uncertainty at date 2, financial contracts between dates 1 and 2 are uncontingent. The resulting Euler equation is

$$m_2^\omega \lambda_1^{i,\omega} = \beta \lambda_2^{i,\omega} + \kappa_2^{i,\omega} \Phi_{2x}^{i,\omega} \quad (12)$$

where  $\Phi_{2x}^{i,\omega} := \partial \Phi_2^{i,\omega} / \partial x_2^{i,\omega}$ . For borrowers, the multiplier on the borrowing constraint satisfies  $\kappa_2^{b,\omega} \geq 0$ , and they attach the shadow value  $\kappa_2^{b,\omega} \Phi_{2x}^{b,\omega}$  to the marginal unit of borrowing.

The optimal capital accumulation decision implies

$$q^\omega \lambda_1^{i,\omega} = \beta \lambda_2^{i,\omega} F_2^{i,\omega'}(k_2^{i,\omega}) + \kappa_2^{i,\omega} \Phi_{2k}^{i,\omega} \quad (13)$$

where  $\Phi_{2k}^{i,\omega} := \partial \Phi_2^{i,\omega} / \partial k_2^{i,\omega}$ . If the financial constraint on agent  $i$  is slack, then  $\kappa_2^{i,\omega} = 0$  and the price of capital is simply its marginal value in the hands of agent  $i$  discounted by the market discount factor  $m_2^\omega = \beta \lambda_2^{i,\omega} / \lambda_1^{i,\omega}$ . This always holds for lenders. Borrowers, on the other hand, may be subject to a binding financial constraint. In that case, equations (12) and (13) capture two effects. First, borrowers discount the future payoff of capital more than lenders,  $\beta \lambda_2^{b,\omega} / \lambda_1^{b,\omega} < m_2^\omega$ , which reduces their valuation of capital. This leads to what is commonly referred to as a *fire-sale discount* in the price of capital. Secondly, the term  $\kappa_2^{i,\omega} \Phi_{2k}^{i,\omega}$  reflects the marginal benefit of relaxing the constraint, which increases borrowers' valuation of capital. The premium captured by this term is what is sometimes called the *collateral value* of capital.

In general equilibrium, optimality conditions (12) and (13) define the price of discount bonds  $m_2^\omega(N^\omega, K_1)$  and capital  $q^\omega(N^\omega, K_1)$  as functions of the aggregate state variables. Both prices are generally – but not always – increasing functions of the net worth of each sector in terms of consumption goods  $N^{i,\omega}$ . Formally, we capture this in the following condition on the response of the asset price to sector  $i$  net worth.

8. The multiplier  $\lambda_i^{i,\omega}$  corresponds to the marginal value of wealth for agent  $i$  in a given date/state and satisfies  $\lambda_i^{i,\omega} = u^{i'}(c_i^{i,\omega}) + \eta_i^{i,\omega}$ . If consumption is positive,  $\lambda_i^{i,\omega}$  is identical to the marginal utility of consumption.

**Condition 1. (Asset price increasing in sectoral net worth)** *The price of capital assets is increasing in the net worth of both sectors,*

$$\frac{\partial q^\omega}{\partial N^{i,\omega}} \geq 0 \quad \forall i \in \{b, \ell\}$$

Intuitively, a marginal increase in  $N^{i,\omega}$  corresponds to injecting more date 1 consumption goods into the economy while holding the amount of capital in the economy fixed. This makes capital goods relatively more scarce. The condition states that this increases the price of capital goods, corresponding to a similar notion to the concept of “ordinary goods” in consumer theory.

Condition 1 is not necessary to derive the two main propositions of our paper. However, it is useful to determine the sign of pecuniary externalities. We impose assumptions on primitives that ensure that the condition is satisfied in each of our four applications in the main text, and we demonstrate in Appendix B.1 how to relate the condition to elasticities of utility and production functions in our first application.<sup>9</sup> We also consider the alternate case in two additional applications in the appendix to show that violations of the condition typically go hand in hand with backward-bending demand curves that lead to multiple and locally unstable equilibria.<sup>10</sup>

We analyze next how changes in the sector-wide date 1 state variables of the economy  $N^\omega$  and  $K_1$  affect the welfare of individual agents. Lemma 1 characterizes the properties of the date 1 equilibrium that are relevant for our efficiency analysis.

**Lemma 1. (Uninternalized welfare effects of changes in sector-wide  $N^\omega$  and  $K_1$ )** *The effects of changes in the sector-wide state variables ( $N^\omega, K_1$ ) on agent  $i$ 's indirect utility at date 1 are given by*

$$V_{Nj}^{i,\omega} := \frac{dV^{i,\omega}(\cdot)}{dN^{j,\omega}} = \lambda_1^{i,\omega} \mathcal{D}_{Nj}^{i,\omega} + \kappa_2^{i,\omega} \mathcal{C}_{Nj}^{i,\omega} \quad (14)$$

$$V_{K_1^j}^{i,\omega} := \frac{dV^{i,\omega}(\cdot)}{dK_1^j} = \lambda_1^{i,\omega} \mathcal{D}_{K_1^j}^{i,\omega} + \kappa_2^{i,\omega} \mathcal{C}_{K_1^j}^{i,\omega} \quad (15)$$

where we refer to  $\mathcal{D}_{Nj}^{i,\omega}$  and  $\mathcal{D}_{K_1^j}^{i,\omega}$  as the **distributive effects** of changes in  $N^{j,\omega}$  and  $K_1^j$  for type  $i$  agents

$$\mathcal{D}_{Nj}^{i,\omega} := -\frac{\partial q^\omega}{\partial N^{j,\omega}} \Delta K_2^{i,\omega} - \frac{\partial m_2^\omega}{\partial N^{j,\omega}} X_2^i \quad (16)$$

$$\mathcal{D}_{K_1^j}^{i,\omega} := F_1^{i,\omega'}(K_1^{i,\omega}) \mathcal{D}_{Nj}^{i,\omega} - \left[ \frac{\partial q^\omega}{\partial K_1^j} \Delta K_2^{i,\omega} + \frac{\partial m_2^\omega}{\partial K_1^j} X_2^{i,\omega} \right] \quad (17)$$

and we refer to  $\mathcal{C}_{Nj}^{i,\omega}$  and  $\mathcal{C}_{K_1^j}^{i,\omega}$  as the **collateral effects** of changes in  $N^{j,\omega}$  and  $K_1^j$  for type  $i$  agents

$$\mathcal{C}_{Nj}^{i,\omega} := \frac{\partial \Phi_2^{i,\omega}}{\partial q^\omega} \frac{\partial q^\omega}{\partial N^{j,\omega}} \quad (18)$$

$$\mathcal{C}_{K_1^j}^{i,\omega} := F_1^{i,\omega'}(K_1^{i,\omega}) \mathcal{C}_{Nj}^{i,\omega} + \frac{\partial \Phi_2^{i,\omega}}{\partial q^\omega} \frac{\partial q^\omega}{\partial K_1^j} \quad (19)$$

9. The behavior of prices cannot be easily stated in terms of fundamentals in almost all general equilibrium models. This makes it useful to focus on sufficient statistics, as we do in our approach.

10. Although a full analysis is outside of the scope of this paper, the index theorem results in Chapter 17 of Mas-Colell et al. (1995) suggest that Condition 1 emerges naturally in models with well-behaved equilibria.

As shown in equations (14) and (15), changes in the sector-wide net worth  $N^{j,\omega}$  and capital  $K_1^j$  affect welfare through two distinct mechanisms that occur because changes in  $N^{j,\omega}$  and  $K_1^j$  affect the equilibrium prices  $q^\omega = (N^\omega, K_1)$  and  $m_2^\omega(N^\omega, K_1)$ : distributive effects and collateral effects. The effects of changes in  $N^{j,\omega}$  and  $K_1^j$  on all other equilibrium variables in problem (11) drop out by the envelope theorem.

First, changes in  $N^{j,\omega}$  and  $K_1^j$  affect the equilibrium prices  $q^\omega(N^\omega, K_1)$  and  $m_2^\omega(N^\omega, K_1)$  at which sector  $i$  agents trade capital and bonds. The *distributive effects*  $\mathcal{D}_{N^j}^{i,\omega}$  and  $\mathcal{D}_{K^j}^{i,\omega}$  capture the marginal wealth redistributions to sector  $i$  that result from price changes following a change in the sector-wide net worth  $N^{j,\omega}$  or capital  $K_1^j$ . We use the terminology distributive effects because they are zero-sum across agents on a state-by-state basis. Formally, exploiting market clearing

$$\sum_i \mathcal{D}_{N^j}^{i,\omega} = 0 \quad \text{and} \quad \sum_i \mathcal{D}_{K^j}^{i,\omega} = 0, \quad \forall \omega \quad (20)$$

Second, changes in the equilibrium price  $q^\omega(N^\omega, K_1)$  directly affect the tightness of the financial constraint faced by borrowers. The *collateral effects*  $\mathcal{C}_{N^j}^{i,\omega}$  and  $\mathcal{C}_{K^j}^{i,\omega}$  capture the direct effect of changes in aggregate state variables on the tightness of  $\Phi_2^{i,\omega}(\cdot)$ . Unlike distributive effects, collateral effects are generally not zero-sum across agents.

In a symmetric equilibrium, it must be that  $n^{i,\omega} = N^{i,\omega}$  and  $k_1^i = K_1^i, \forall i$ . In that case, agent  $i$ 's indirect utility is given by  $V^{i,\omega}(N^{i,\omega}, K_1^i; N^\omega, K_1)$ , and we can decompose the equilibrium effects of a change in sector  $i$  financial net worth  $N^{i,\omega}$  on welfare into two parts

$$\frac{dV^{i,\omega}(N^{i,\omega}, K_1^i; N^\omega, K_1)}{dN^{i,\omega}} = V_n^{i,\omega}(\cdot) + V_{N^i}^{i,\omega}(\cdot)$$

The term  $V_n^{i,\omega} := \partial V^{i,\omega} / \partial n^{i,\omega}$  represents the private marginal utility of wealth and is given by the envelope condition  $V_n^{i,\omega}(\cdot) = \lambda_1^{i,\omega}$ . This part is internalized by individual agents who choose how much wealth to carry into date 1. The term  $V_{N^i}^{i,\omega}$  represents the effects of changes in sector-wide net worth that are *not internalized* by individual agents. A similar decomposition can be performed for the internalized and uninternalized effects of changes in sector-wide capital  $k_1^i = K_1^i$ . In our welfare analysis in Section 3, these uninternalized effects will represent pecuniary externalities.

*Date 0 equilibrium.* We describe the date 0 optimization problem of agent  $i$  as

$$\max_{c_0^i \geq 0, k_1^i, x_1^{i,\omega}} u^i(c_0^i) + \beta \mathbb{E}_0 \left[ V^{i,\omega} \left( e_1^{i,\omega} + x_1^{i,\omega} + F_1^{i,\omega}(k_1^i), k_1^i; N^\omega, K_1 \right) \right] \quad \text{s.t.} \quad (2), (5) \quad (21)$$

Using the envelope conditions  $V_n^{i,\omega}(\cdot) = \lambda_1^{i,\omega}$  and  $V_k^{i,\omega}(\cdot) = \lambda_1^{i,\omega} q^\omega$ , we obtain a set of standard Euler equations and an optimal investment condition

$$m_1^\omega \lambda_0^i = \beta \lambda_1^{i,\omega} + \kappa_1^i \Phi_{1x^\omega}^i, \quad \forall i, \omega \quad (22)$$

$$h^{i'}(k_1^i) \lambda_0^i = \mathbb{E}_0 \left[ \beta \lambda_1^{i,\omega} \left( F_1^{i,\omega'}(k_1^{i,\omega}) + q^\omega \right) \right] + \kappa_1^i \Phi_{1k}^i, \quad \forall i \quad (23)$$

where we define  $\Phi_{1x^\omega}^i := \partial \Phi_1^i / \partial x_1^{i,\omega}$  and  $\Phi_{1k}^i := \partial \Phi_1^i / \partial k_1^i$ , we assign  $\kappa_1^b$  as the (vector) multiplier on the financial constraint of borrowers and define  $\kappa_1^\ell := 0$  for lenders to keep notation symmetric.

The Euler equations ensure that the intertemporal marginal rates of substitution of all agents are equated to the market prices  $m_1^\omega$  and thus to each other in every state of nature,

unless the financial constraint introduces a wedge. The optimal investment condition states that the marginal cost of capital investment equals its discounted marginal benefit, which consists of the marginal product  $F_1^{i,\omega}(k_1^{i,\omega})$ , the continuation value  $q^\omega$  of capital, and the benefit of relaxing the constraint.

### 3. EFFICIENCY ANALYSIS

We set up a constrained social planner problem in the tradition of Stiglitz (1982) and Geanakoplos and Polemarchakis (1986) to determine if the decentralized equilibrium is constrained efficient. The social planner chooses date 0 allocations subject to the same constraints as the private market, leaving all later decisions to private agents, and respecting that capital and security prices are market-determined.<sup>11</sup>

Formally, the constrained social planner maximizes the weighted sum of welfare of the two sets of agents for given Pareto weights  $(\theta^b, \theta^\ell)$ . The planner chooses date 0 allocations  $(C_0^i, K_1^i, X_1^{i,\omega})$ , subject to the date 0 resource constraint. To emphasize that the planner chooses sector-wide variables, we denote her allocations by upper-case letters. Given our earlier definition of date 1 indirect utility functions  $V^{i,\omega}(\cdot)$ , the constrained planner's problem is

$$\begin{aligned} \max_{C_0^i, K_1^i, X_1^{i,\omega}} \sum_i \theta^i \left\{ u^i(C_0^i) + \beta \mathbb{E}_0 \left[ V^{i,\omega}(N^{i,\omega}, K_1^i; N^\omega, K_1) \right] \right\} & \quad (24) \\ \text{s.t.} \quad \sum_i \left[ C_0^i + h^i(K_1^i) - e_0^i \right] \leq 0 & \quad (v_0) \\ \sum_i X_1^{i,\omega} = 0, \forall \omega & \quad (v_1^\omega) \\ \Phi_1^i(X_1^i, K_1^i) \geq 0, \forall i & \quad (\theta^i \kappa_1^i) \\ C_0^i \geq 0, \forall i & \quad (\theta^i \eta_0^i) \end{aligned}$$

where  $N^{i,\omega} = e_1^i + X_1^{i,\omega} + F_1^{i,\omega}(K_1^i)$ ,  $\forall \omega, i$ . We assign the shadow price  $v_0$  to the date 0 resource constraint,  $v_1^\omega$  to the intertemporal resource constraint for state  $\omega$ , the vector of shadow prices  $\theta^i \kappa_1^i$  to the financial constraint, and  $\theta^i \eta_t^{i,\omega}$  to the multipliers on the non-negativity constraints of consumption.<sup>12</sup> We also denote the marginal value of wealth for agent  $i$  by  $\lambda_t^{i,\omega} = u^{i'}(C_t^{i,\omega}) + \eta_t^{i,\omega}$  – it equals the marginal utility of consumption except when consumption is at a corner solution.

Proposition 1 characterizes constrained efficient allocations and shows how to implement them. Proposition 2 identifies the two distinct externalities that underlie inefficiency and establishes that each of them can be characterized as a function of a small set of variables that determine their sign and magnitude.

**Proposition 1. a) (Constrained efficient allocations)** *A date 0 allocation  $(C_0^i, K_1^i, X_1^{i,\omega})$  is constrained efficient if and only if there are positive welfare weights that satisfy  $\theta^b/\theta^\ell = \lambda_0^\ell/\lambda_0^b$  and shadow prices  $v_0, v_1^\omega$ , and  $\kappa_1^i$  such that the allocation respects the constraints in problem (24) and satisfies*

11. This setup is equivalent to the problem of a constrained Ramsey planner who chooses taxes on date 0 allocations plus transfers, as shown in the online appendix.

12. We scale all agent-specific multiplier by  $\theta^i$  to keep notation symmetric with the optimization problem of private agents.

the following financing and investment conditions

$$\frac{v_1^\omega}{v_0} \lambda_0^i = \beta \lambda_1^{i,\omega} + \kappa_1^i \Phi_{1x^\omega}^i + \beta \sum_{j \in I} \frac{\theta^j}{\theta^i} V_{N^i}^{j,\omega}, \quad \forall i, \omega \quad (25)$$

$$h^{i'}(K_1^i) \lambda_0^i = \beta \mathbb{E}_0 \left[ \lambda_1^{i,\omega} \left( F_1^{i,\omega'}(K_1^i) + q^\omega \right) \right] + \kappa_1^i \Phi_{1k}^i + \beta \sum_{j \in I} \frac{\theta^j}{\theta^i} \mathbb{E}_0 \left[ F_1^{i,\omega'}(K_1^i) V_{N^i}^{j,\omega} + V_{K^i}^{j,\omega} \right], \quad \forall i \quad (26)$$

where all variables at dates 1 and 2 are determined by the optimization problem (11) and market clearing, and  $V_{N^i}^{j,\omega}$  and  $V_{K^i}^{j,\omega}$  are defined in Lemma 1.

**b) (Implementing constrained efficiency)** A planner can implement any constrained efficient allocation by setting taxes on state-contingent security purchases and capital investment that satisfy

$$\tau_x^{i,\omega} = - \sum_{j \in I} MRS^{j,\omega} \mathcal{D}_{N^i}^{j,\omega} - \sum_{j \in I} \tilde{\kappa}_2^{j,\omega} \mathcal{C}_{N^i}^{j,\omega}, \quad \forall i, \omega \quad (27)$$

$$\tau_k^i = - \sum_{j \in I} \mathbb{E}_0 \left[ MRS^{j,\omega} \mathcal{D}_{K^i}^{j,\omega} \right] - \sum_{j \in I} \mathbb{E}_0 \left[ \tilde{\kappa}_2^{j,\omega} \mathcal{C}_{K^i}^{j,\omega} \right], \quad \forall i \quad (28)$$

where  $MRS^{j,\omega} := \beta \lambda_1^{j,\omega} / \lambda_0^j$  and  $\tilde{\kappa}_2^{j,\omega} := \kappa_2^{j,\omega} / \lambda_0^j$ , and conducting lump-sum transfers  $T^i$  such that date 0 budget constraints (2) with taxes are met and the government budget constraint  $\sum_i T^i = \sum_i \mathbb{E}_0[\tau_x^{i,\omega} X_1^{i,\omega}] + \sum_i \tau_k^i K_1^i$  is satisfied.

Proposition 1.a) characterizes constrained efficient allocations through a set of Euler equations for financing and investment decisions, as in the decentralized case. The left hand side of equation (25) is the social marginal price of saving one unit of wealth. The right hand side is the associated social marginal benefit, consisting of the consumption value of an extra unit of net worth, the value of relaxing the financial constraint and the uninternalized welfare effects described in Lemma 1. Similarly, the left hand side of (26) reflects the social marginal cost of capital investment, and the social marginal benefits on the right hand side consist of the marginal product of capital, the continuation value of capital  $q^\omega$ , the benefits of capital in relaxing the financial constraint, and the uninternalized welfare effects described in Lemma 1. A comparison of equations (22) and (23) with equations (25) and (26) highlights that the sole difference between the decentralized and the constrained efficient allocation is that the planner internalizes the general equilibrium effects captured by these uninternalized welfare effects. This difference corresponds to the taxes described in equations (27) and (28) of Proposition 1.b).

Proposition 1.b) describes how to set the corrective tax instruments  $\tau_x^{i,\omega}$  and  $\tau_k^i$  to modify agents' date 0 decisions and implement constrained efficient allocations. Intuitively, these tax rates induce private agents to internalize the pecuniary externalities of their actions caused by both the distributive and collateral effects. A positive  $\tau_x^{i,\omega}$  induces agent  $i$  to allocate fewer resources towards state  $\omega$  – indicating that private agents underborrow in the decentralized equilibrium; a positive  $\tau_k^i$  induces agent  $i$  to invest less in capital – indicating that private agents overinvest in the decentralized equilibrium – and vice versa for negative signs. As the proposition illustrates, optimal corrective policies are agent-specific and cannot in general be implemented as an anonymous set of taxes.<sup>13</sup>

13. In general, when the planner is constrained to use anonymous linear taxes, the optimal corrective policy is given by a cross-sectional weighted average of the individual taxes  $\tau_x^{i,\omega}$  and  $\tau_k^i$  identified in equations (27) and (28), following the logic of Diamond (1973). When the allocations of agents differ sufficiently, a non-linear anonymous tax schedule that imposes different rates on borrowing and lending may also be able to replicate the optimal tax system.

Proposition 1 holds verbatim for more than two types of agents, as we show in the appendix. In the two-agent case, we can simplify the tax rates (27) and (28). For the first additive term in each expression, corresponding to the distributive effects, we exploit market clearing, as in equation (20), and define  $\Delta MRS^{ij,\omega} := MRS^{i,\omega} - MRS^{j,\omega}$  as the difference between marginal rates of substitution between agents. For the second term, corresponding to the collateral effects, we simply note that  $\tilde{\kappa}_2^{\ell,\omega} = 0$  by construction. This allows us to re-write equations (27) and (28) and express  $\tau_x^{i,\omega}$  and  $\tau_k^i$  as follows:

$$\tau_x^{i,\omega} = -\Delta MRS^{ij,\omega} \mathcal{D}_{N^i}^{i,\omega} - \tilde{\kappa}_2^{b,\omega} \mathcal{C}_{N^i}^{b,\omega}, \quad \forall i, \omega \quad (29)$$

$$\tau_k^i = -\mathbb{E}_0 \left[ \Delta MRS^{ij,\omega} \mathcal{D}_{K^i}^{i,\omega} \right] - \mathbb{E}_0 \left[ \tilde{\kappa}_2^{b,\omega} \mathcal{C}_{K^i}^{b,\omega} \right], \quad \forall i \quad (30)$$

Proposition 2 formally establishes the distinct nature of distributive and collateral externalities. For both types of externalities, the direction of the inefficiency is fully determined by a small set of sufficient statistics with a natural interpretation.

**Proposition 2. (Distinct nature of externalities/sufficient statistics)** *There are two distinct types of externalities: distributive externalities (D) and collateral externalities (C).*

*The sign and magnitude of distributive externalities are determined by the product of three variables:*

- (D1) *The difference in MRS of agents  $\Delta MRS^{ij,\omega}$*
- (D2) *The net trading positions (net buying or net selling) on capital  $\Delta K_2^{i,\omega}$  and financial assets  $X_2^{i,\omega}$*
- (D3) *The sensitivity of equilibrium prices to changes in sector-wide state variables  $\frac{\partial q^\omega}{\partial N^{i,\omega}}, \frac{\partial m_2^\omega}{\partial N^{i,\omega}}, \frac{\partial q^\omega}{\partial K_1^i}, \frac{\partial m_2^\omega}{\partial K_1^i}$*

*The sign and magnitude of collateral externalities are determined by the product of three variables:*

- (C1) *The shadow value on the financial constraint  $\tilde{\kappa}_2^{i,\omega}$*
- (C2) *The sensitivity of the financial constraint to the asset price  $\partial \Phi_2^{i,\omega} / \partial q^\omega$*
- (C3) *The sensitivity of the equilibrium capital price to changes in sector-wide state variables  $\frac{\partial q^\omega}{\partial N^{i,\omega}}, \frac{\partial q^\omega}{\partial K_1^i}$*

Proposition 2 contains one of the main economic insights of this paper. A small number of sufficient statistics encapsulate the information needed to determine whether an economy is constrained efficient and how to correct any distortions. Distributive and collateral externalities are generically present in any competitive environment in which financial market imperfections nest into the form of equations (5) and (6).

Distributive externalities arise because agents do not internalize that their actions change equilibrium prices, affecting the amount received by other agents through capital or financial asset sales or purchases. When financial constraints inhibit optimal risk-sharing and prevent the equalization of MRS between agents across dates or states, independently of the reason why MRS are not equalized, a suitable change in the behavior of agents redistributes resources through price changes towards agents with higher MRS in a given date/state, improving efficiency.

Therefore, understanding the nature of distributive externalities requires to understand the difference in relative valuations of wealth (i.e. the MRS) of all agents across dates/states, their net trading positions, and how changes in sector-wide state variables affect equilibrium prices.

Collateral externalities arise because agents do not internalize that their actions change equilibrium prices, directly modifying the borrowing/saving capacity of other constrained

agents. A suitable change in the behavior of agents modifies asset prices, relaxing financial constraints directly and changing the effective financial decisions of those agents for which the constraint binds.

Therefore, understanding the nature of collateral externalities requires to understand the welfare benefit of relaxing borrowers' financial constraint, the change in borrowing capacity due to a change in asset prices, and the sensitivity of equilibrium prices to changes in sector-wide state variables.

While distributive externalities operate by changing the value of the flow of resources, collateral externalities operate by directly affecting the financing capacity of constrained agents by changing the value of the stock of assets that serve as collateral, not just the flow of resources between agents. For this reason, only borrowers in our baseline model experience the effects of collateral externalities, while all agents experience the effects of distributive externalities.

As usual in normative problems, it is in general not feasible to characterize distortions or optimal corrective policies as a function of primitives.<sup>14</sup> Instead, Proposition 2 shows that, independently of the specific nature of the financial frictions, identifying the sign and magnitude of the externalities boils down to identifying a small number of sufficient statistics, which should guide the design of corrective policies. These variables will be invariant to the precise nature of the underlying distortions, e.g., uncontingent bonds, limited commitment, market segmentation, idiosyncratic risks, etc. In Section 4, we illustrate in specific applications how changes in primitives affect the sign and magnitude of the externalities through changes in these sufficient statistics. The sufficient statistics that we identify in Proposition 2 remain the key determinants of the sign of the externalities in more general environments with multiple agents and more general preferences and production technologies, as shown in the online appendix.<sup>15</sup>

Propositions 1 and 2 characterize the entire Pareto frontier of the economy as a planner varies the relative welfare weights  $\theta^l / \theta^b$  on the two types of agents. When the decentralized equilibrium is constrained inefficient, there is a continuum of constrained efficient allocations that constitute Pareto improvements, which we characterize formally in Corollary 6 in the online appendix. Each of these constrained efficient allocations corresponds to different relative welfare weights and requires different lump-sum transfers and optimal tax rates to be implemented.<sup>16</sup> However, an additional advantage of our approach is that the optimal taxes are fully determined by the sufficient statistics and depend on the welfare weights only indirectly.

An important application of our optimal tax formulas is to identify general circumstances under which equilibria with financially constrained agents and fire sales are *constrained efficient*. Distributive pecuniary externalities are zero whenever either (i) financially constrained agents face complete risk markets to insure against future fire sales so  $\Delta MRS^{bl,\omega} = 0$  or (ii) the net trading position of capital and financial assets is zero or (iii) the prices of capital and financial assets are fixed, e.g. because of linear preferences and technologies. We will show an example of (i) below in Application 1, and examples of (ii) and (iii) below in Application 4. Collateral externalities are absent whenever (i) borrowers are unconstrained at date 1 so  $\kappa_2^{b,\omega} = 0$  or (ii) their financial constraint only depends on individual-level variables so  $\partial \Phi_2^{b,\omega} / \partial q^\omega = 0$  or (iii)

14. Even the most elementary results in normative economics are expressed as a function of high level observables as opposed to primitives. For instance, Ramsey's characterization of optimal commodity taxes relies on demand elasticities, which are endogenous to the level of taxes.

15. The online appendix also considers more general constraint sets  $\Phi_i^j(\cdot)$  that depend directly on aggregate state variables, e.g., moral-hazard/incentive constraints or value-at-risk requirements. For further examples see Greenwald and Stiglitz (1986). We show that these are of the same nature as collateral externalities, although it may be more appropriate to call them binding-constraint externalities instead of collateral externalities. In the appendix, we explain how to adjust the sufficient statistics for collateral effects to this more general case.

16. In fact, even the sign of the optimal taxes may depend on the chosen welfare weights.

the prices of capital assets are fixed. When neither type of externality is present, optimal taxes are zero and equilibrium is constrained efficient.

In the following corollaries, we provide five general results that follow from our analysis. We further elaborate on those results in our applications in Section 4.

*Sign of externalities.* In the existing literature on pecuniary externalities, it has proven remarkably difficult to provide general results on the direction of inefficiency – except in tightly-defined special cases. The following corollary rationalizes why.

**Corollary 1. (Sign of externalities and “anything goes”)** *The collateral externalities of sector-wide net worth are non-negative under Condition 1. All distributive externalities as well as the collateral externalities of sector-wide capital holdings can naturally take on either sign, so “anything goes.”*

The corollary states that in general, only the collateral externalities of financing decisions can be signed since the sufficient statistics  $C1$  and  $C2$  are by construction non-negative; the shadow value of borrowers’ financial constraint is weakly positive and a higher asset price weakly relaxes the financial constraint. Furthermore,  $C3$  is positive for sector-wide net worth if and only if the natural Condition 1 is satisfied, implying that collateral externalities unambiguously lead to overborrowing in that case.

The sufficient statistics  $D1$  and  $D2$  can naturally take on either sign; plausible configurations of primitives are consistent with positive or negative differences in MRS and with agents that can be net buyers or sellers. For example, if borrowers have a high relative valuation compared to lenders in a given state and they are net sellers of capital in that state, it will be optimal to subsidize their savings towards that state. Furthermore, the sufficient statistics  $C3$  and  $D3$  can take on either sign for the externalities of sector-wide capital holdings. As a result, “anything goes” for the sign of distributive externalities and the collateral externalities of sector-wide capital holdings.

Unpacking the optimal tax rates for distributive and collateral externalities into three sufficient statistics each is also helpful in spelling out explicit conditions under which they can be signed. This is useful if we are explicitly concerned with devising conditions under which the direction of inefficiency can be pinned down unambiguously, as we demonstrate in the applications in Section 4 and as a number of papers that we discuss in Section 5 have done.

*Externality pricing kernel.* To apply Proposition 1 to a broader set of financial assets, consider a financial security  $Z$  that is traded at date 0 with a state-contingent payoff profile  $(Z^\omega)_{\omega \in \Omega}$  at date 1. This security can be viewed as a bundle of Arrow securities with weight  $Z^\omega$  on the security contingent on state  $\omega \in \Omega$ . For example, a risk-free bond corresponds to  $Z^\omega = 1, \forall \omega$ . To hold constant the set of trading opportunities, we require that total security holdings satisfy  $\tilde{x}_1^{i,\omega} = x_1^{i,\omega} + \alpha_Z Z^\omega$ , where  $\Phi_1^i(\tilde{x}_1^{i,\omega}, k_1^i) \geq 0$  and  $\alpha_Z$  denotes the holdings of security  $Z$ . Under this assumption, the security  $Z$  is redundant and no-arbitrage pricing implies Corollary 2.

**Corollary 2. (Externality pricing kernel)** *The optimal corrective tax on agent  $i$ ’s holdings of a financial security  $Z$  is given by*

$$\tau_Z^i = \mathbb{E}_0 \left[ \tau_x^{i,\omega} Z^\omega \right], \quad (31)$$

where  $\tau_x^{i,\omega}$  is given by equation (27).

Equation (31) reveals a close parallel between traditional security pricing and the pricing of pecuniary externalities. We can view the optimal state-contingent tax rates  $\tau_x^{i,\omega}$  defined in Proposition 1 as an *externality pricing kernel*, which is used to determine the social cost of sector  $i$  holding a security with payoff profile  $Z^\omega$ . The corollary provides a simple expression to guide financial regulators on the design of optimal corrective policies for any financial instrument.

*Investment and financing distortion.* Since investing in capital and financial assets are both mechanisms for shifting resources across time, Corollary 2 also implies a relation between the distortions in investment and financing decisions. Increasing capital  $K_1^i$  in sector  $i$  has two general equilibrium effects. First, it increases output and sector  $i$ 's net worth by  $F_1^{i,\omega'}(K_1^i)$  at date 1. In that sense, increasing  $K_1^i$  is identical to saving  $F_1^{i,\omega'}(K_1^i)$  by sector  $i$  while holding  $N^{j,\omega}$  constant for the other sector. Secondly, additional capital increases output at date 2, which has general equilibrium effects on prices  $q^\omega$  and  $m_2^\omega$  and may in turn lead to distributive and collateral effects, as described in Lemma 1. The following corollary describes the relationship when the latter effects are absent.

**Corollary 3. (Relationship between distortion in investment and financing decisions)** *When  $\partial q^\omega / \partial K_1^i = \partial m_2^\omega / \partial K_1^i = 0$ , the optimal corrective taxes  $\tau_x^{i,\omega}$  and  $\tau_k^i$  on financing and investment decisions satisfy*

$$\tau_k^i = \mathbb{E}_0 \left[ \tau_x^{i,\omega} F_1^{i,\omega'}(K_1^i) \right] \quad (32)$$

Corollary 3 implies that an optimal policy must coordinate the distortions introduced in investment and saving decisions. When  $\partial q^\omega / \partial K_1^i = \partial m_2^\omega / \partial K_1^i = 0$ , the resulting relationship is simple and intuitive. The condition holds for example when production and utility functions are linear at date 2. In that case, both distortions are tightly linked. The general case in which  $\partial m_2^\omega / \partial K_1^i$  and  $\partial q^\omega / \partial K_1^i$  can take any values is formally described in the appendix.

*Fire sales, amplification and welfare.* The role of fire sales, amplification effects, and their relation with efficiency of an economy are often intertwined in policy discussions. However, fire sales and amplification both describe positive phenomena happening in the economy, which do not necessarily have normative implications that lead to inefficiency. Although there are no universally agreed-upon definitions of the two concepts, it is useful to define them as follows in the context of our framework.<sup>17</sup>

We define fire sales as instances when financially constrained agents sell capital assets at a price that discounts the future returns that they could earn at a higher rate than the market discount rate. Furthermore, it is natural to define financial amplification as instances when a decline in sector-wide net worth  $N^{i,\omega}$  leads to general equilibrium effects that deteriorate the sector's terms of trade or that tighten the financial constraint on the sector, and vice versa for positive shocks. Exploiting the definitions from Lemma 1, these two situations

17. The insights that emerge from Corollary 4 do not depend on the precise definitions adopted. More broadly, Corollary 4 implies that positive phenomena, like positive feedback among variables, or price changes, are neither necessary nor sufficient to obtain constrained inefficiency.

formally correspond to amplification via distributive effects, which occurs when  $\mathcal{D}_{Ni}^{i,\omega} > 0$ , and amplification via collateral effects, which occurs when  $\mathcal{C}_{Ni}^{i,\omega} > 0$ .<sup>18,19</sup>

**Corollary 4. (Decoupling of fire sales, amplification and inefficiency)** *The existence of fire sales or of amplification effects ( $\mathcal{D}_{Ni}^{i,\omega} > 0$  or  $\mathcal{C}_{Ni}^{i,\omega} > 0$ ) is neither necessary nor sufficient for constrained inefficiency.*

Fire sales are not necessary because inefficient pecuniary externalities may also arise when constrained agents are buyers of capital or do not trade capital. Amplification effects are not necessary because constrained inefficiency can also arise when  $\mathcal{D}_{Ni}^{i,\omega} < 0$ . Neither fire sales nor amplification effects are sufficient because there are several situations in which the two phenomena are consistent with constrained efficiency. First, if there are fire sales and amplification via distributive effects  $\mathcal{D}_{Ni}^{i,\omega} > 0$  but decentralized agents face complete date 0 financial markets and equate their MRS, Proposition 1 implies that equilibrium is constrained efficient. Second, if there are fire sales and amplification but decentralized agents are in a corner solution, a planner may not be able to improve welfare, implying that equilibrium is constrained efficient.

Even when amplification leads to constrained inefficiency, Corollary 1 implies that the sign of the resulting distortion is indeterminate – amplification via distributive effects  $\mathcal{D}_{Ni}^{i,\omega} > 0$  may be consistent with both over- and underborrowing.

*Indeterminacy and simplified implementation results.* Proposition 1 provides the most transparent exposition of pecuniary externalities in that it attributes taxes to each decision margin and each sector according to the externalities it creates. However, intervening in every single decision margin of private agents imposes a significant burden on regulators. In fact, we show that the planner faces up to three degrees of freedom in the choice of a constrained optimal tax system. This allows us to normalize one or more of the policy instruments  $\{\tau_x^{b,\omega}, \tau_x^{\ell,\omega}, \tau_k^b, \tau_k^\ell\}$  to zero or to impose anonymous instead of agent-specific taxes.<sup>20</sup> In the following, we assume that all tax changes are performed in a wealth-neutral manner, i.e. any additional tax revenue raised from a given sector is returned to the same sector in the form of a lump-sum transfer.

**Corollary 5. (Degrees of freedom in setting taxes/simplified implementation)** *a) There are up to three degrees of freedom in setting taxes to implement a given constrained optimal allocation: (1) we can change the tax burden on borrowers vs. lenders in any state  $\omega$  by varying  $\tau_x^{i,\omega}$  and  $m_1^\omega$  such that the sum  $(\tau_x^{i,\omega} + m_1^\omega)$  remains unchanged for each agent  $i \in I$ ; (2) if consumption is a corner solution ( $\eta_0^i > 0$ ), we can change the tax burden on financing vs. investment decisions for any agent  $i$  by jointly varying  $\tau_x^{i,\omega}$ ,  $\tau_k^i$  and letting  $\eta_0^i$  adjust; (3) if the financial constraint is binding ( $\kappa_1^b(z) > 0$  for the  $z$ 'th element*

18. These definitions capture the typical notion in the literature on financial amplification that shocks at the sector-wide level are amplified via general equilibrium effects and have greater effects than the identical shocks on an individual agent. In the notation of Lemma 1, this is captured by the inequality  $V_n^i + V_{Ni}^i > V_n^i$ . The general equilibrium effects in turn induce the affected sector to reduce consumption and/or asset purchases in response to an adverse shock by more than if the shock affected an individual agent.

19. For amplification via distributive effects, an alternative definition would be to focus on the distributive effects of moving a marginal unit of net worth from type  $j$  to type  $i$ , corresponding to  $\mathcal{D}_{Ni}^{i,\omega} - \mathcal{D}_{Nj}^{i,\omega} > 0$  in our notation. Our corollary below still applies.

20. This finding is also useful to relate our optimal tax formulas to the existing literature. For example, there are a number of papers in which constrained efficiency can be achieved by taxing financing decisions only, even though both financing and investment decisions are distorted, as we will discuss in further detail in Section 5.

of the constraint function  $\Phi_1^b$ ), we can change the tax burden on financing vs. investment decisions for borrowers by jointly varying both  $\tau_k^b$  (if  $\Phi_{1k}^b(z) > 0$ ) and  $\tau_x^{b,\omega}$  (for all  $\omega$  for which  $\Phi_{1x\omega}^b(z) > 0$ ) and letting the shadow price  $\kappa_1^b(z)$  adjust.

b) These degrees of freedom allow us in case (1) to normalize  $\tau_x^{i,\omega} = 0$  for one of the agents  $\forall \omega \in \Omega$ . In cases (2) and (3) they allow us to normalize either  $\tau_k^i = 0$  or  $\tau_x^{i,\omega} = 0$  for one  $\omega \in \Omega$  or, alternatively, to impose anonymous taxes  $\tau_k^i = \tau_k, \forall i$  or  $\tau_x^{i,\omega} = \tau_x^\omega, \forall i$  for one  $\omega \in \Omega$  (as long as the respective constraints are sufficiently binding).

The first degree of freedom, or indeterminacy, captures that agents only care about the after-tax price of financial securities when they trade – a parallel change in the tax rates on borrowers and lenders moves the pre-tax market prices  $m_1^\omega$  but does not affect the after-tax prices ( $\tau_x^{i,\omega} + m_1^\omega$ ) faced by each sector. It also leaves the total wedge between borrowers and lenders ( $\tau_x^{b,\omega} - \tau_x^{\ell,\omega}$ ) unchanged. Assuming that the tax change is performed in a wealth-neutral manner, the resulting allocation is unaffected. This indeterminacy allows a financial regulator to impose taxes or regulation on the financing decisions of one sector (e.g. lenders) and leave the financing behavior of the other sector (e.g. borrowers) unregulated.

The second and third degrees of freedom, or indeterminacies of implementation, arise when either date 0 consumption is a corner solution (i.e. the non-negativity constraint on  $c_0^i$  is binding,  $\eta_0^i > 0$ ) or when the date 0 financial constraint is binding. In both cases, agents effectively face a single decision margin between borrowing and investing. Both  $\tau_x^{i,\omega}$  and  $\tau_k^i$  target that single decision margin and can substitute for each other. In particular, it is sufficient for a regulator to regulate only the financing of sector  $i$  and leave investment decisions unregulated. We provide an example in the appendix. Naturally, all three of the described degrees of freedom/strategies for simplifying implementation can be combined.

When private agents face corner solutions, an important implication of the corollary is that any decision margin on which the taxes can be set to zero can be interpreted as constrained efficient. For example, in an economy in which all financing and investment decisions are fully determined by binding constraints, the three indeterminacies together imply that we can set  $\tau_x^{i,\omega} = \tau_k^i = 0 \forall i$  – a planner cannot improve on the decentralized allocation if there are no free decision margins – and equilibrium is constrained efficient. Another important application is that the corollary may allow us to impose anonymous taxes when the decisions of one of the two agents are corner solutions or determined by a binding constraint. For example, if lenders do not produce capital  $k_1^\ell = 0$ , then an anonymous tax can correct the investment decisions of borrowers.<sup>21</sup>

#### 4. APPLICATIONS

We present four specific applications that allow us to zero in on the efficiency results of Proposition 1 and illustrate how the sufficient statistics that underlie the sign of pecuniary externalities may easily flip sign, as described in Proposition 2 and Corollary 1.

Application 1 describes a setting in which there are fire sales but the economy is constrained efficient since risk markets are complete and financial constraints do not depend on prices. Applications 2 and 3 provide two distinct examples in which the sign of distributive externalities depends on the primitives of the model. In Application 2, we describe a setting

21. If the planner has additional instruments, further degrees of freedom arise. For example, a tax on borrowing and subsidy to investment can be substituted by a tax on consumption, reflecting that when there is both over-borrowing and under-investment, there must be over-consumption.

in which borrowers switch from being net buyers to being net sellers of capital when their endowment crosses a threshold, which changes the sign of sufficient statistic  $D2$  and therefore the sign of the inefficiency. In Application 3, we describe a setting in which borrowers may be either constrained in their borrowing or in their saving, depending on their initial endowment. As their endowment crosses the relevant thresholds, the difference in the MRS of borrowers and lenders and, by implication, the sufficient statistic  $D1$  and the associated distributive externalities change sign. Application 4 provides an example of collateral externalities and illustrates that this always leads to overborrowing, but may be consistent with either under- or over-investment.

Below, we keep preferences and endowments unrestricted in Application 1 to illustrate the generality of the efficient fire-sale phenomenon. In Applications 2 to 4, we assume that lenders have linear utility  $U^\ell = c_0^\ell + c_1^\ell + c_2^\ell$  and large endowments of the consumption good at each date, which pins down  $m_2^\omega \equiv 1$  so there are no distributive effects from bond prices, and that borrowers have no endowments at dates 1 and 2, that is,  $e_1^b = e_2^b = 0$ . Throughout all four applications, we assume that borrowers have an investment technology  $h^b(k) = \alpha k^2/2$ ; we also assume that lenders cannot create capital at date 0, corresponding to  $h^\ell(k) = \infty$  for any positive  $k > 0$ , except in Application 2, in which lenders have the same investment technology as borrowers so we can study trade of capital in both directions and changes in the sign of  $D2$ . We also assume throughout our applications that borrowers have a more productive use for capital than lenders, except in Application 3, in which the production function of lenders satisfies an Inada condition so they always purchase some capital, pinning down the sign of  $D2$  and allowing us to focus on changes in the sign of  $D1$ . Regarding financial frictions, in Applications 1 and 4, we assume that the date 0 market for Arrow securities is complete, captured by  $\Phi_1^b \equiv 0$ , so that no inefficient distributive externalities arise; in Applications 2 and 3 we assume that the date 1 financial market is w.l.o.g. completely shut down,  $x_2^{b,\omega} = 0, \forall \omega$ , which is captured by the constraint specification  $\Phi_2^{b,\omega}(x_2^{b,\omega}) = -(x_2^{b,\omega})^2 \geq 0$ , to obtain simple analytic results for the distributive externalities stemming from the date 0 financial imperfections.

In each application, we illustrate our results graphically with a single figure. The parameter values used to draw the figures are described in the appendix.

#### 4.1. Efficient Fire Sales

*Environment.* We build on the baseline model from Section 2 and assume a specific formulation for production technologies and financial constraints. Formally, the production technology of borrowers is linear  $F_t^{b,\omega}(k) = A_t^\omega k$ , whereas that of lenders is equally productive for the first marginal unit  $F_t^{\ell,\omega'}(0) = A_t^\omega$  but exhibits strictly decreasing returns  $F_t^{\ell,\omega''}(k) < 0$ . Date 1 productivity  $A_1^\omega$  is a random variable, with negative realizations representing reinvestment requirements, and date 2 productivity is given by a constant  $A_2 > 0$ .

Regarding financial constraints, at date 0, borrowers face complete financial markets but, at date 1, they can only pledge to repay at most a fraction  $\phi$  of their date 2 production. Formally, the constraint they face at date 1 is

$$\Phi_2^{b,\omega}(x_2^{b,\omega}, k_2^{b,\omega}) := x_2^{b,\omega} + \phi F_2^{b,\omega}(k_2^{b,\omega}) \quad \forall \omega$$

*Equilibrium and efficiency.* It is simplest to illustrate our results graphically. Figure 1 shows date 1 equilibrium prices  $m_2^\omega$  and  $q^\omega$  as well as saving  $X_2^{b,\omega}$  and capital holdings  $(K_2^{b,\omega}, K_2^{\ell,\omega})$  as a function of borrowers' net worth  $N^{b,\omega}$  for given  $K_1^b$  and  $N^{\ell,\omega}$ . We describe the general conditions

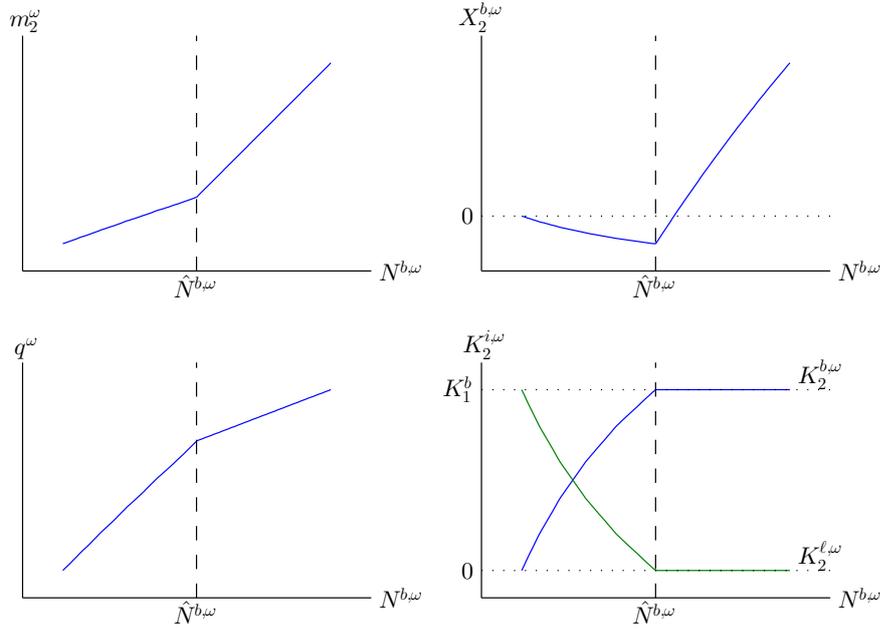


FIGURE 1  
Date 1 Equilibrium

under which the date 1 equilibrium of the economy is well-defined and unique as well as the specific parameters used in the figure in the appendix.

Figure 1 captures both optimal smoothing, when the financial constraint is slack, and fire sales, when the constraint is binding. For any pair  $(K_1^b, N^{\ell,\omega})$ , we can define a threshold  $\hat{N}^{b,\omega} = \hat{N}^b(K_1^b, N^{\ell,\omega})$  such that, for  $N^{b,\omega} \geq \hat{N}^{b,\omega}$ , borrowers keep all capital and save a fraction of any additional net worth to smooth consumption between dates 1 and 2. Bond prices and capital prices increase (interest rates decrease) in parallel with borrowers' net worth, reflecting the greater abundance of wealth at date 1. If borrowers' net worth falls below the threshold,  $N^{b,\omega} < \hat{N}^{b,\omega}$ , the borrowing constraint binds, and borrowers sell some of their capital, which forces them to reduce their borrowing. The price functions  $m_2^\omega$  and  $q^\omega$  experience a kink at the threshold because the rate at which borrowers exchange date 1 and date 2 consumption goods with lenders becomes more disadvantageous when fire sales are involved.

At date 0, agents make investment and financing decisions optimally by solving problem (21). If these decisions lead to  $N^{b,\omega} \geq \hat{N}^{b,\omega}$  for all states  $\omega$ , then the financial constraint is always slack and the allocation is first-best. Otherwise, the financial constraint binds and fire sales occur in some states. Independently of whether fire sales occur at date 1 or not, we find the following result:

**Application 1. (Efficient fire sales).** *The decentralized equilibrium in the described economy is constrained efficient.*

The general lesson of this application is that when agents face complete financial markets between dates 0 and 1, the welfare effects of distributive externalities cancel in the decentralized

equilibrium. Formally, choosing Pareto weights  $\theta^i = 1/u^{i'}(c_0^i)$ , it is the case that

$$\sum_i \theta^i \lambda_1^{i,\omega} \mathcal{D}_{N^i}^{i,\omega} = 0 \quad \forall \omega, j \quad (33)$$

and similar for the distributive effects of capital holdings.

Equation (33) is stronger than our earlier observation in equation (20) that distributive externalities are zero-sum,  $\sum_i \mathcal{D}_{N^i}^{i,\omega} = 0$ , at every date and state. It instead shows that we can find welfare weights such that the welfare impact of the distributive externalities nets out to zero. Because complete financial markets allow agents to equalize their marginal valuations of wealth (MRS) across all states of nature, there is no scope for the planner to increase efficiency by using distributive effects.

This application highlights that in practice, we do not need to be concerned with the distributive effects of fire sales as long as the agents engaging in the transactions have optimally shared risk in complete insurance markets. This is likely the case e.g. for sophisticated financial sector participants trading with each other. Conversely, if financial sector participants are forced to sell to outsiders who have not participated in optimal ex-ante risk-sharing, distributive externalities are of concern.

#### 4.2. Distributive Externalities and Direction of Capital Trade

*Environment.* We assume that both borrowers and lenders have linear utility  $U^i = c_0^i + c_1^i + c_2^i$ , with  $c_t^i \geq 0$ , and access to the date 0 investment technology given by  $h(k) = \alpha k^2/2$ . Borrowers have no endowment. Borrowers' production function is linear  $F_t^b(k) = A_t^\omega k$  with  $A_t^\omega > 0$ , while that of lenders is the same  $F_1^\ell(k) = A_1^\omega k$  at date 1, but takes the value  $F_2^\ell(k) = A_2 \log(1+k)$  at date 2. A binary shock  $\omega \in \Omega = \{L, H\}$  that affects solely productivity  $A_1^\omega$  is realized at date 1. In the first-best, borrowers and lenders invest such that  $h'(k^i) = \alpha k^i = \mathbb{E}[A_1^\omega] + A_2$  or  $k_1^i = (\mathbb{E}[A_1^\omega] + A_2)/\alpha$  for  $i = b, \ell$ . Since borrowers have the more efficient production technology at date 2, they hold all the capital, so  $k_2^{b,\omega} = 2k_1^b$  and  $k_2^{\ell,\omega} = 0$ .

Regarding financial constraints, we assume that only uncontingent bonds are available for trade at date 0 and that no borrowing or lending is possible at date 1, capturing a shutdown of financial markets. Formally, borrowers' date 0 constraint is given by  $\Phi_1^b(x_1)^b := (x_1^{b,L} - x_1^{b,H}) = 0$ , while their date 1 constraint is given by  $\Phi_2^{b,\omega}(x_2^{b,\omega}) := x_2^{b,\omega} = 0$ .

*Equilibrium and efficiency.* The date 1 demand for capital assets by lenders is given by their optimality condition

$$q^\omega = F_2^{\ell'}(k_2^{\ell,\omega}) = \frac{A_2}{1 + k_2^{\ell,\omega}} \quad (34)$$

Given borrowers' net worth  $n^{b,\omega} = A_1^\omega k_1^b + x_1^b$ , their date 1 budget constraint, financial constraint, and non-negativity constraint on consumption can be combined into

$$n^{b,\omega} + q^\omega (k_1^b - k_2^{b,\omega}) \geq 0 \quad (35)$$

Therefore, borrowers' date 1 value function is

$$V^b(n^{b,\omega}, k_1^b; N^\omega, K_1) = \max_{k_2^{b,\omega}} A_2 k_2^{b,\omega} + \lambda_1^{b,\omega} [n^{b,\omega} + q^\omega (k_1^b - k_2^{b,\omega})] \quad (36)$$

We define the threshold  $\hat{N}^b := A_2 K_1^\ell$  and observe that the date 1 financial constraint is slack if  $N^{b,\omega} \geq \hat{N}^b$  and binding otherwise. If the constraint is slack, borrowers buy up all capital in the economy  $K_2^{b,\omega} = K_1^b + K_1^\ell$  at a price  $q^\omega = V_k^{b,\omega} = A_2$ , which is independent of sectoral net worth so all distributive effects are zero,  $\mathcal{D}_{N^i}^{i,\omega} = \mathcal{D}_{K^j}^{i,\omega} = 0$ . The marginal value of date 1 borrower wealth in that case is  $\lambda_1^{b,\omega} = V_n^{b,\omega} = 1$ .

If  $N^{b,\omega} < \hat{N}^b$ , borrowers' financial constraint binds, causing them to reduce their capital holdings below the efficient level,  $K_2^{b,\omega} < K_1^b + K_1^\ell$ . Combining lenders' demand (34) with borrowers' constraint (35) yields the equilibrium capital holdings and price of capital

$$K_2^{b,\omega} = K_1^b + \frac{N^{b,\omega}(1 + K_1^\ell)}{N^{b,\omega} + A_2} \quad \text{and} \quad q^\omega = \frac{N^{b,\omega} + A_2}{1 + K_1^\ell}$$

where a well-defined equilibrium with strictly positive capital holdings exists when  $N^{b,\omega} \geq N^{b,\min} = -A_2 K_1^b / (1 + K_1^b + K_1^\ell)$ . The equilibrium price of capital depends exclusively on two aggregate state variables  $q^\omega(N^{b,\omega}, K_1^\ell)$  and satisfies  $\partial q^\omega / \partial N^{b,\omega} > 0$  and  $\partial q^\omega / \partial K_1^\ell < 0$ . The resulting distributive effects, as defined in Lemma 1, are

$$\begin{aligned} \mathcal{D}_{N^b}^{b,\omega} &= -\frac{\partial q^\omega}{\partial N^{b,\omega}} \Delta K_2^{b,\omega} = -\frac{N^{b,\omega}}{N^{b,\omega} + A_2} \\ \mathcal{D}_{K^b}^{b,\omega} &= A_1^\omega \mathcal{D}_{N^b}^{b,\omega} = -\frac{A_1^\omega N^{b,\omega}}{N^{b,\omega} + A_2} \\ \mathcal{D}_{K^\ell}^{b,\omega} &= -\frac{\partial q^\omega}{\partial K_1^\ell} \Delta K_2^{b,\omega} = \frac{N^{b,\omega}}{1 + K_1^\ell} \end{aligned}$$

Lenders' net worth does not have distributive effects,  $\mathcal{D}_{N^\ell}^{i,\omega} = 0$ . When the financial constraint binds, the signs of the distributive externality terms can take all possible values, depending on the value of  $N^{b,\omega}$ . We distinguish three regions for the date 1 equilibrium. First, if  $N^{b,\omega} \in [N^{b,\min}, 0)$ , borrowers fire-sell assets, so  $K_2^{b,\omega} < K_1^b$ . In this region, higher borrowers' net worth and lower lenders' capital raise the price at which constrained borrowers are forced to sell their capital, which distributes resources towards borrowers. Therefore, the distributive effects of additional borrowers' net worth or capital are positive  $\mathcal{D}_{N^b}^{b,\omega}, \mathcal{D}_{K^b}^{b,\omega} > 0$ , while those of additional lenders' capital are negative  $\mathcal{D}_{K^\ell}^{b,\omega} < 0$ . Second, if  $N^{b,\omega} = 0$ , borrowers neither buy nor sell additional capital so  $K_2^{b,\omega} = K_1^b$ , even though the marginal products satisfy  $F_2^{\ell\ell}(K_2^{\ell,\omega}) < F_2^{b\ell}(K_2^{b,\omega})$ . In this knife-edge case there is no trade in capital goods and all distributive effects are zero,  $\mathcal{D}_{N^b}^{b,\omega} = \mathcal{D}_{K^b}^{b,\omega} = \mathcal{D}_{K^\ell}^{b,\omega} = 0$ . Third, if  $N^{b,\omega} \in (0, \hat{N}^b)$ , borrowers purchase additional capital but less than the optimal amount so  $K_1^b < K_2^{b,\omega} < K_1^b + K_1^\ell$ . In this region, the distributive effects of additional borrowers' net worth or capital are negative  $\mathcal{D}_{N^b}^{b,\omega}, \mathcal{D}_{K^b}^{b,\omega} < 0$  but those of additional lenders' capital are positive  $\mathcal{D}_{K^\ell}^{b,\omega} > 0$ .

We provide a full characterization of the date 0 equilibrium in the appendix. The interesting case is when the financial constraint is slack in the high state and binding in the low state. In that case, distributive effects have the following efficiency implications.

**Application 2. (Changing sign of capital trade).** *There is a threshold value  $\tilde{A}_1^L$  such that a) if  $A_1^L < \tilde{A}_1^L$ , the economy exhibits overborrowing and underinvestment by borrowers as well as overinvestment by lenders, b) if  $A_1^L = \tilde{A}_1^L$ , the economy is constrained efficient, and c) if  $A_1^L > \tilde{A}_1^L$ ,*

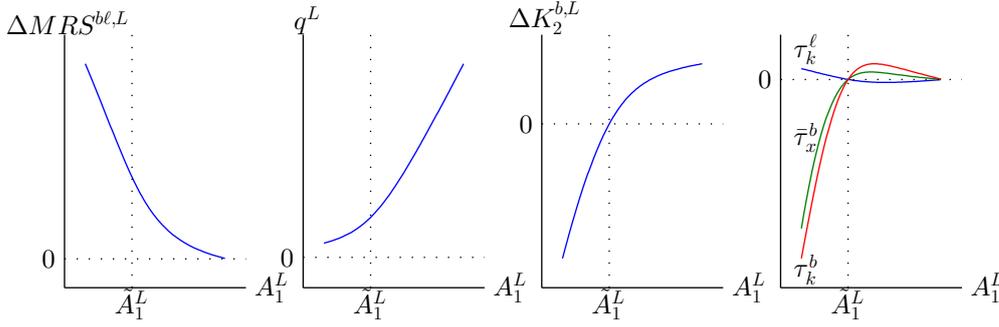


FIGURE 2

Components of Optimal Taxes  $\bar{\tau}_x^b$ ,  $\tau_k^\ell$ , and  $\tau_k^b$  in Application 2

the economy exhibits underborrowing and overinvestment by borrowers as well as underinvestment by lenders.

Intuitively, case a) represents a traditional scenario in which constrained borrowers fire-sell assets: a marginal reduction in borrowing, or additional date 1 capital income, pushes up the price that borrowers fetch for their fire sales of capital. Since borrowers are constrained in the low state and were unable to arrange contingent insurance towards that state, the positive distributive effects, captured by  $\mathcal{D}_{N^b}^{b,\omega}, \mathcal{D}_{K^b}^{b,\omega} > 0$ , improve insurance. A marginal reduction in investment by lenders also raises the fire-sale price and has similar effects, captured by  $\mathcal{D}_{K^\ell}^{b,\omega} < 0$ . Case b) corresponds to the knife-edge case in which borrowers are constrained but neither buy nor sell assets; as a result, marginal changes to either borrowing or investment do not affect welfare. In case c), borrowers are also constrained but have positive date 1 net worth, which allows them to purchase some assets from lenders. Higher borrowing and lower investment by borrowers reduce borrowers' net worth and push down the price of capital, making it cheaper for constrained borrowers to buy assets, captured by  $\mathcal{D}_{N^b}^{b,\omega}, \mathcal{D}_{K^b}^{b,\omega} < 0$ . Similarly, more investment by lenders reduces the price of capital, implying  $\mathcal{D}_{K^\ell}^{b,\omega} > 0$ .<sup>22</sup>

Figure 2 depicts the key variables that drive the sign and magnitude of the distributive externalities as we vary  $A_1^L$ . The first panel shows the difference in MRS between agents in the low state of nature, which is decreasing in  $A_1^L$  and always weakly positive in state  $L$  since borrowers are weakly constrained. The second panel depicts the price of capital in the low state  $q^L$ , which is increasing in the shock realization  $A_1^L$  as higher income implies smaller fire sales. The third panel shows borrowers' net trading position of capital  $\Delta K_2^{b,L}$  in the low state, which changes sign at the productivity threshold  $\bar{A}_1^L$ . The fourth panel illustrates the resulting tax rates: for  $A_1^L < \bar{A}_1^L$ , it is optimal to tax borrowing (i.e. subsidize saving so  $\bar{\tau}_x^b < 0$ ), subsidize investment by borrowers (so  $\tau_k^b < 0$ ) and tax investment by lenders (so  $\tau_k^\ell > 0$ ), and vice versa for  $A_1^L > \bar{A}_1^L$ . We denote by  $\bar{\tau}_x^b$  the optimal tax on uncontingent bond holdings since the date 0 financial constraint implies that only uncontingent bonds can be traded.

The general lesson of this application is that constrained borrowers may be either buyers or sellers of capital and financial assets. They may switch from one to the other in response to small changes in fundamentals. As a result, sufficient statistic  $D2$  can take on either sign.

22. Whether the distortions to investment of borrowers and lenders imply over- or under-investment in aggregate depends on parameter values and cannot be determined in general.

It is straightforward to enrich this application allowing for multiple states of nature in which borrowers are constrained, some of which satisfy  $A_1^\omega < \tilde{A}_1^L$  and others  $A_1^\omega > \tilde{A}_1^L$ . In that case, there will be overborrowing towards some states and underborrowing towards others.

In practice, such behavior is common for arbitrageurs and market makers who trade with agents in incomplete risk markets. If they become financially constrained in response to an adverse shock but have sufficient net worth so they can still buy assets, then their net worth generates negative distributive externalities; if the shock is large enough that they are forced to sell, then their net worth generates positive distributive externalities.

#### 4.3. Distributive Externalities and Sign of $\Delta MRS$

*Environment.* In this application, we assume a perfect foresight economy with no uncertainty. Borrowers have concave utility  $U^b = \sum_{t=0}^2 \log(c_t^b)$  and a non-negative date 0 endowment  $e_0^b \geq 0$  that we vary as our main experiment. Both borrowers and lenders are unproductive at date 1, but produce according to  $F_2^b(k) = Ak$  and  $F_2^\ell(k) = A \log(k)$  at date 2, where  $A > \alpha$ . In the first-best, borrowers' date 0 investment corresponds to  $k_1^b = \frac{A}{\alpha}$ . At date 1, capital holdings are  $k_2^\ell = 1$  and  $k_2^b = k_1^b - 1$ . Borrowers' consumption is equalized across all dates.

Regarding financial constraints, we assume that borrowers face limits both on how much they can save and how much they can borrow at date 0 due to limited commitment, so  $\phi \leq x_1^b \leq \bar{\phi}$ . As in Application 2, no borrowing or lending is possible at date 1. Formally, borrowers' date 0 financial constraint is given by the vector inequality

$$\Phi_1^b(x_1^b) := \begin{pmatrix} \bar{\phi} - x_1^b \\ x_1^b - \phi \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

while borrowers' date 1 financial constraint is given by  $\Phi_2^b(x_2^b) := x_2^b = 0$ .

*Equilibrium and efficiency.* The date 1 demand for capital assets by lenders is given by their optimality condition

$$q = F_2^{\ell'}(k_2^\ell) = \frac{A}{k_2^\ell} \quad (37)$$

Given borrowers' date 1 net worth  $n^b = x_1^b$ , their date 1 budget constraint and financial constraint can be combined to  $c_1^b = n^b + q(k_1^b - k_2^b)$ . Therefore, their date 1 value function is

$$V^b(n^b, k_1^b; N, K_1) = \max_{k_2^b} u^b(n^b + q(k_1^b - k_2^b)) + u^b(Ak_2^b), \quad (38)$$

Their optimality condition for capital holdings is given by

$$q = \frac{n^b}{2k_2^b - k_1^b} \quad (39)$$

Combining lenders' demand (37) for capital with borrowers' supply (39) yields the equilibrium capital holdings and price of capital

$$K_2^b = \frac{A + N^b}{2A + N^b} K_1^b \quad \text{and} \quad q = \frac{2A + N^b}{K_1^b}$$

where a well-defined equilibrium with non-negative capital holdings exist when  $N^b \geq -A$ . Consequently, the date 1 equilibrium level of consumption for borrowers corresponds to  $C_1^b =$

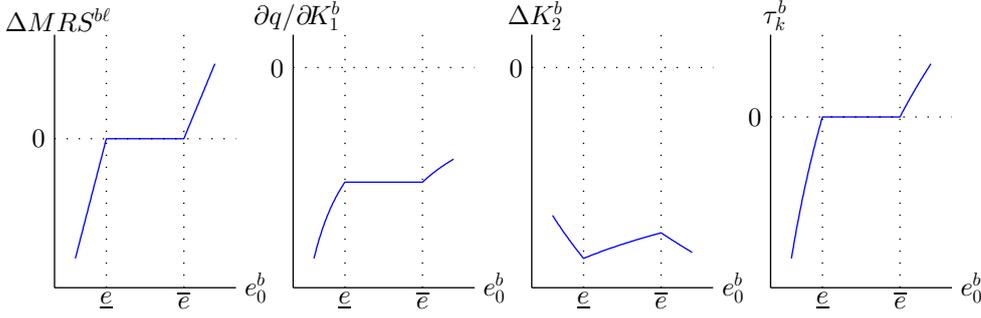


FIGURE 3

Components of Optimal Tax  $\tau_k^b$  in Application 3

$N^b + q(K_1^b - K_2^b) = N^b + A$ . Borrowers receive a constant amount  $A$  for their sales of capital, independently of the number of units of sold.

The equilibrium asset price depends exclusively on two aggregate state variables  $q(N^b, K_1^b)$  and satisfies  $\partial q / \partial N^b > 0$  and  $\partial q / \partial K_1^b < 0$ . The distributive effects in this economy, as defined in Lemma 1, satisfy

$$\mathcal{D}_{N^b}^b = -\frac{\partial q}{\partial N^b} \Delta K_2^b > 0$$

$$\mathcal{D}_{K^b}^b = -\frac{\partial q}{\partial K_1^b} \Delta K_2^b < 0$$

where we use the fact that borrowers are always net sellers of capital, that is,  $\Delta K_2^b = -\frac{A}{2A+N^b} < 0$  is negative. As in our previous application, lenders' net worth does not have distributive effects, so  $\mathcal{D}_{N^\ell}^i = 0$ . Since lenders do not have capital at date 0, characterizing  $\mathcal{D}_{K^\ell}^i$  is irrelevant.

We provide a full characterization of the date 0 equilibrium in the appendix. We show that the optimal unconstrained financial decision by borrowers is given by

$$X_1^* = \frac{e_0^b - 2A}{2.5}$$

Therefore, depending on the borrowers' endowment  $e_0^b$ , the date 0 equilibrium can take three different forms. First, when  $X_1^* \in [\phi, \bar{\phi}]$ , the financial constraint is slack, so  $X_1^b = X_1^*$ . Second, when  $X_1^* < \phi$ , then borrowers hit their borrowing limit, so  $X_1^b = \phi$ . Third, when  $X_1^* > \bar{\phi}$ , then borrowers hit their saving limit, so  $X_1^b = \bar{\phi}$ .

Because borrowers are constrained in their choice of  $x_1^b$  whenever the financial constraint is binding, Corollary 5 implies that the same financing decision  $x_1^b$  can be implemented by setting  $\tau^b = 0$ . Borrowing is therefore always constrained efficient in this economy, and it is sufficient to focus on the optimal corrective policy for investment  $\tau_k^b$ . Out of the three sufficient statistics from Proposition 2, our application restricts the signs of  $D2$  and  $D3$ , since borrowers are always net sellers and asset prices increase with the level of  $K_1^b$ . We summarize our findings on how the form of the date 0 equilibrium affects  $D1$  and efficiency of capital investment as follows.

**Application 3. (Changing sign of  $\Delta MRS$ )** There are two thresholds  $e$  and  $\bar{e}$  for the value of the difference in borrowers' endowments  $e_0^b$  such that a) if  $e_0^b < e$ , then  $\Delta MRS^{b\ell} < 0$ , and the economy exhibits under-investment, b) if  $e \leq e_0^b \leq \bar{e}$ , the economy is constrained efficient, and c) if  $e_0^b > \bar{e}$ , then

$\Delta MRS^{bl} > 0$ , and the economy exhibits over-investment. Borrowing decisions are always constrained efficient.

In case a), when  $e_0^b < \underline{e}$ , borrowers hit their date 0 borrowing limit, and  $\Delta MRS^{bl} < 0$  – they value wealth at date 1 relative to date 0 less than lenders. This makes it desirable to allocate more wealth to lenders at date 1 and more to borrowers at date 0, but the borrowing constraint prevents the financial market from performing this operation. However, the planner increases capital investment by borrowers, which reduces the price at which borrowers sell capital at date 1 and effectively redistributes resources to lenders at date 1. Moreover, the planner provides a lump-sum transfer from lenders to borrowers at date 0. Taken together, these two interventions constitute a second-best way for the planner to emulate the effects of borrowing at date 0. Conversely, in case c), borrowers would like to save more wealth than the saving limit allows for and  $\Delta MRS^{bl} > 0$ . A planner can circumvent this constraint through reduced capital investment by borrowers, which increases the date 1 price of capital and redistributes resources to borrowers, combined with a lump-sum transfer from borrowers to lenders at date 0.

Figure 3 depicts the key variables that drive the sign and magnitude of the distributive externalities as we vary the endowment parameter  $e_0^b$ . The first panel illustrates that  $\Delta MRS^{bl}$  is weakly increasing over the entire domain of  $e_0^b$  – it is negative for  $e_0^b < \underline{e}$  and positive for  $e_0^b > \bar{e}$ . The second and third panels depict the response of the asset price to capital  $\partial q / \partial K_1^b$  and borrowers' net trading position of capital  $\Delta K_2^b$ , both of which are always negative. The fourth panel combines the three sufficient statistics from the first three panels and reports the resulting tax rate  $\tau_K^b$ , which is negative for low  $e_0^b$  (implying a subsidy) and positive for high  $e_0^b$ .

The general lesson of this application is that the relative intertemporal valuation of resources by agents – captured by  $\Delta MRS^{bl}$  – can take on either sign, and borrowers may switch from having a higher valuation of resources to having a lower valuation of resources in response to changes in fundamentals. We have set the described application in perfect foresight to crystallize the message that distributive effects arise even if there is just a single state of nature. In that case, a planner can only improve efficiency if she also has access to date 0 lump-sum transfers so that the combination of distributive effect plus transfer substitute for incomplete date 0 financial markets – without transfers, the distributive effects would constitute mere movements along a constrained Pareto frontier. For an elaboration of this point, see Dávila (2014). In more general stochastic settings, incomplete risk markets give rise to differences in marginal rates of substitution for different agents across multiple states of nature, and the planner can employ distributive effects to improve efficiency by improving risk-sharing between agents, even when lump-sum transfers are ruled out.

This application captures two different scenarios in practice. On the one hand, agents may be constrained in their saving, as observed by a branch of literature that studies shortages of safe assets (see e.g. Caballero and Farhi, 2014). On the other hand, borrowers may hit their borrowing limit. Our application suggests that in such situations, a policymaker could improve welfare by taxing the agents with excess savings or subsidizing the constrained agents, and manipulating their capital investment decision to affect their terms-of-trade in the future in an offsetting manner.

#### 4.4. Collateral Externalities

*Environment.* Our last application illustrates the workings of collateral externalities. We assume that borrowers have quasilinear utility  $U^b = \log c_0^b + \log c_1^b + c_2^b$  with  $c_2^b \geq 0$  as well as a date 0 endowment  $e_0^b \in [0, 1]$ . This is a perfect foresight economy with no uncertainty.

Borrowers have a linear production technology while lenders have no use for capital. Formally,  $F_t^b(k) = A_t k$  and  $F_t^\ell(k) = 0$  for  $t = 1, 2$  where we assume  $0 < A_1 + A_2 \leq \alpha$ . Importantly, we assume  $A_2 > 0$ , but we do not restrict the sign of  $A_1$  in this application – a negative value may capture a reinvestment requirement that is necessary to maintain the capital. Our assumptions on utility and the technology of lenders imply that the bond price  $m_2 = 1$  is constant and capital is not traded among sectors, so all distributive effects are zero  $\mathcal{D}_{N^j}^i = \mathcal{D}_{K^b}^i = 0$ , focusing our analysis exclusively on collateral externalities. In the first-best, as shown in the appendix,  $C_0^b = C_1^b = 1$ ,  $K_t^{b*} = (A_1 + A_2)/\alpha$  and  $q = A_2$ .

Regarding financial constraints, we assume that borrowers are unconstrained at date 0. We also assume that borrowers can only borrow up to a fraction  $\phi$  of the value of their asset holdings at date 1, where  $\phi \in (0, 1/(1+A_2))$  to ensure equilibrium is well-defined. Formally,  $\Phi_1^b := 0$  and

$$\Phi_2^b(x_2^b, k_2^b; q) := x_2^b + \phi q k_2^b \geq 0 \quad (40)$$

*Equilibrium and efficiency.* Since capital always remains in the hands of borrowers, the price of capital is pinned down by borrowers' optimality condition for capital holdings

$$q = \frac{A_2}{u^{br}(c_1^b) + \phi \kappa_2^b} = \frac{A_2 c_1^b}{1 - \phi + \phi c_1^b} \quad (41)$$

which defines a function  $q(C_1^b)$  with  $q'(C_1^b) > 0$ .

The interesting case is when the financial constraint (40) is binding. As we show in the appendix, this is the case if  $N^b \in (0, 1 - \phi A_2 K_1^b)$ . When borrowing is constrained to  $X_2^b = -\phi q K_1^b$ , the date 1 budget constraint implies that

$$C_1^b = N^b + \phi q K_1^b = N^b + \phi K_1^b \frac{A_2 C_1^b}{1 - \phi + \phi C_1^b} \quad (42)$$

This equation defines a unique level of consumption  $C_1^b(N^b, K_1^b)$  since the right-hand side is increasing in  $C_1^b$  at a slope of less than one, given that our assumption on  $\phi$  implies  $\phi/(1 - \phi) \cdot A_2 < 1$  and given that  $K_1^b \leq K_1^{b*} \leq 1$  in equilibrium. Consumption is increasing in both  $N^b$  and  $K_1^b$ . Substituting this consumption level into (41), the price of capital is a function  $q(N^b, K_1^b) = q(C_1^b(N^b, K_1^b))$  that is also increasing in both  $N^b$  and  $K_1^b$ . Equilibrium is independent of  $N^\ell$ , and  $K_t^\ell = 0$  at all times.

Since the collateral constraint (40) depends on the price of capital, changes in the sector-wide state variables  $(N^b, K_1^b)$  have collateral effects, as defined in Lemma 1

$$\begin{aligned} C_{N^b}^b &= \phi K_1^b \frac{\partial q}{\partial N^b} > 0 \\ C_{K_1^b}^b &= \phi K_1^b \left( A_1 \frac{\partial q}{\partial N^b} + \frac{\partial q}{\partial K_1^b} \right) = \frac{\phi K_1^b q'(C_1^b)}{1 - \phi K_1^b q'(C_1^b)} \cdot (A_1 + \phi q) \gtrless 0 \end{aligned}$$

The sign of the collateral effects of capital depends on the sign of the last term in parentheses.  $A_1$  is the marginal date 1 payoff of capital, which may be negative, and  $\phi q$  is the borrowing capacity generated by an additional unit of capital. If  $A_1 < -\phi q$ , then additional capital reduces the liquid net worth of the borrower sector at date 1, which reduces  $q$  and generates negative collateral effects, and vice versa for  $A_1 > -\phi q$ .

As shown in the appendix, the date 0 equilibrium is determined by the private Euler equation  $C_0^b = C_1^b$  and the following optimality condition for capital investment, which equates

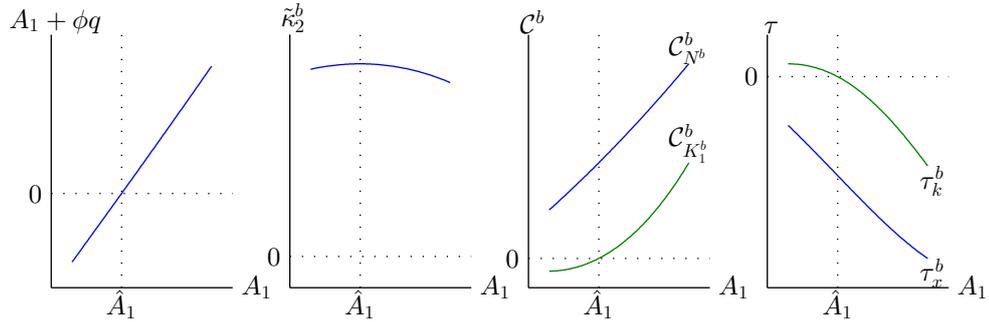


FIGURE 4  
Components of Optimal Taxes  $\tau_x^b, \tau_k^b$  in Application 4

the marginal cost of investment to its private marginal benefit

$$h^{b'}(k_1^b) = 2k_1^b = A_1 + q \quad (43)$$

For our main experiment, we vary the parameter  $A_1$  and express all equilibrium variables as a function of  $A_1$ . We define the threshold  $\hat{A}_1$  such that  $\hat{A}_1 + \phi q(\hat{A}_1) = 0$ , where  $q(A_1)$  maps the equilibrium price to the level of  $A_1$ . The normative properties of the described economy when the financial constraint is binding are as follows.

**Application 4. (Collateral externalities).** Borrowers engage in overborrowing. There is a threshold  $\hat{A}_1 < 0$  such that borrowers a) over-invest if  $A_1 < \hat{A}_1$ , b) engage in efficient investment if  $A_1 = \hat{A}_1$  and c) under-invest if  $A_1 > \hat{A}_1$ .

Our result on overborrowing is a simple application of Corollary 1 and corresponds to the sign of  $C_{N^b}^b$ , which is unambiguous. The distortion on borrowers' level of investment depends on whether the marginal date 1 payoff plus the borrowing capacity generated by additional capital is positive or negative. If  $\hat{A}_1 + \phi q(\hat{A}_1) < 0$ , then additional capital soaks up liquidity in period 1 and there is over-investment; in the converse case additional capital generates net liquidity and there is underinvestment.<sup>23</sup>

Figure 4 depicts the key variables that drive the magnitude of the collateral externalities as we vary the date 1 payoff of capital  $A_1$ . The first panel shows the net liquidity generated by additional capital  $A_1 + \phi q$ , which is increasing in  $A_1$ . The second panel depicts the shadow price on the date 1 collateral constraint, which is hump-shaped – at first it increases since higher  $A_1$  makes borrowers more eager to invest; then it declines as higher  $A_1$  generates more date 1 liquidity, which relaxes the constraint. The third panel shows the collateral effects as defined in Lemma 1:  $C_{N^b}^b$  is always positive when the constraint binds whereas  $C_{K^b}^b$  changes sign when the threshold  $\hat{A}_1$  is crossed. The fourth panel reports the resulting tax wedges on bonds and capital investment, which represent the product of the variables shown in panels 2 and 3. It is always desirable to tax borrowing when the constraint is binding, whereas the sign of the optimal tax on capital switches at  $\hat{A}_1$ .

This application illustrates the result of Corollary 1 that collateral externalities lead to overborrowing. Since asset prices are increasing in sectoral net worth and since higher asset

23. Note that when there is too much borrowing and too little investment at date 0, borrowers are consuming too much.

prices relax collateral constraints, it is desirable for a planner to induce private agents to save more when faced with collateral externalities, i.e. there is overborrowing in the decentralized equilibrium. Furthermore, the application illustrates that collateral externalities may lead to either under- or over-investment.

In the real world, financial constraints that depend on asset prices are pervasive, both among financial sector participants who are subject to margin constraints, and in the real economy, for example among households purchasing homes or firms investing in real assets. This application illustrates that in all these situations, it is desirable to shore up the net worth of the agents determining the relevant asset prices. This can be done by inducing agents to buy insurance against those states of nature in which prices decline or, absent insurance markets, to restrict their borrowing. Furthermore, it is desirable to intervene in their investment decisions to mitigate price declines. If the assets in question drain liquidity when financial constraints are binding, e.g. houses that need to be maintained or highly cyclical capital investment that generates losses during recessions, then it is desirable to restrict investment in them; if the assets in question provide additional liquidity, e.g. by paying reliable dividends and providing a safe stream of income, then it is desirable to induce additional investment.

## 5. RELATED LITERATURE

Our paper is part of a strand of literature that analyzes pecuniary externalities in settings with fire sales and financial amplification. Hart (1975) and Stiglitz (1982) were the first to identify pecuniary externalities that give rise to inefficiency when financial markets are incomplete in the sense that the set of available assets is exogenously limited. Geanakoplos and Polemarchakis (1986) generalized their results and showed that competitive equilibrium is generically constrained inefficient in such a setting. This inefficiency is the basis of what we call distributive externalities: changes in allocations influence market prices in a way that improves risk-sharing or intertemporal smoothing. Greenwald and Stiglitz (1986) showed that pecuniary externalities also arise when private agents are subject to other constraints that depend on market prices such as selection or incentive constraints. This inefficiency is closely related to what we call collateral externalities: changes in allocations influence markets prices in a way that relaxes binding price-dependent constraints.<sup>24</sup>

The analysis of financial amplification and fire sale effects as positive phenomena dates back to at least Fisher (1933) and includes seminal contributions by Bernanke and Gertler (1990), Shleifer and Vishny (1992), and Kiyotaki and Moore (1997). See Krishnamurthy (2010), Shleifer and Vishny (2011), and Brunnermeier and Oehmke (2013) for recent surveys. The main observation in these works is that changes in the net worth of borrowers may be amplified by price changes that further reduce their net worth, corresponding to distributive amplification effects (Shleifer and Vishny, 1992), or that tighten binding constraints on borrowers, corresponding to collateral amplification effects (cf. Corollary 4). The remainder of this section relates the existing literature on financial amplification and pecuniary externalities to our framework with the objective of highlighting the precise mechanisms that lead to inefficiency through the lens of our sufficient statistics results.

Gromb and Vayanos (2002) analyze financially constrained agents who arbitrage between segmented markets in an environment with incomplete risk markets (because market

24. Prescott and Townsend (1984) show how such pecuniary externalities can be overcome when agents can directly contract consumption levels and no anonymous re-trading is allowed. In a similar vein, Kilenthong and Townsend (2014) propose to create segregated security exchanges with entry fees/subsidies for the exclusive right and obligation to trade in a particular exchange, representing a Coasian solution to restore efficiency in pecuniary externality problems.

segmentation restricts the set of assets that investors other than arbitrageurs can trade) and price-dependent collateral constraints (which limit the borrowing of arbitrageurs). This gives rise to both distributive and collateral externalities in our terminology. As a result, borrowing and risk-taking by arbitrageurs can be either excessive or, when distributive externalities dominate, insufficient.

Caballero and Krishnamurthy (2003) analyze risk-taking by emerging market agents who insure against aggregate shocks but face uninsurable idiosyncratic shocks. Retrading among domestic agents after the idiosyncratic shock is realized generates distributive externalities that lead to excessive aggregate risk-taking and overinvestment. Lorenzoni (2008) considers fire sales in an economy in which borrowers are financially constrained and limited commitment by lenders constrains insurance provision, similar to Application 3. This generates distributive externalities that give rise to overinvestment and excessive borrowing against the good state of nature. In both papers, efficiency can be restored by solely taxing borrowing, even though both borrowing and investment decisions are distorted. This is an application of our Corollary 5.

Distributive externalities also arise in the literature that studies liquidity provision and the coexistence of financial intermediaries and markets since the possibility of spot retrading in financial markets, together with market incompleteness, reduces risk sharing opportunities. See for example Jacklin (1987), Bhattacharya and Gale (1987), Allen and Gale (2004) and Farhi et al. (2009). Kehoe and Levine (1993) endogenize financial constraints from limited commitment and exclusion from intertemporal markets. Rampini and Viswanathan (2010) endogenously derive state-dependent collateral constraints through limited commitment without exclusion. Other recent papers that consider distributive externalities include Uribe (2006), Dávila et al. (2012), Hart and Zingales (2015) and He and Kondor (2016).

Jeanne and Korinek (2010a,b) and Bianchi and Mendoza (2012) consider borrowers in a dynamic setting who are subject to a price-dependent collateral constraint that introduces collateral externalities which lead to excessive borrowing, as implied by our Corollary 1. They assume that lenders cannot hold capital and have linear utility so no distributive effects arise, similar to Application 4. Other recent papers in which prices or other aggregate state variables enter into financial constraints, generating externalities of the same nature as the collateral externalities studied in our paper, include Gersbach and Rochet (2012), Stein (2012), Benigno et al. (2013) and Kilenthong and Townsend (2014).

There is also a complementary strand of literature that focuses on aggregate demand externalities in the presence of nominal price rigidities. See e.g. Farhi and Werning (2016), Korinek and Simsek (2016) and Schmitt-Grohé and Uribe (2016). These externalities are qualitatively different from the pecuniary externalities that we study. Farhi and Werning (2016) provide an integrated welfare analysis of both aggregate demand and pecuniary externalities. Korinek and Simsek (2016) illustrate that the two types of externalities interact and may mutually reinforce each other.

## 6. CONCLUSION

This paper develops a general framework to characterize the pecuniary externalities that arise in economies with financially constrained agents. We identify two distinct externalities, distributive and collateral externalities, and show that each of the two types can be quantified as a function of three intuitive sufficient statistics. Distributive externalities occur when a planner can employ changes in prices to allocate wealth to agents who are underinsured because of incomplete insurance markets. Collateral externalities occur when a planner can employ changes in prices to relax binding collateral constraints that depend on market prices.

Incomplete insurance markets and financial constraints that depend on prices are pervasive in both theory and practice, suggesting that our findings have broad applicability.

Our general framework and our dichotomy between distributive and collateral externalities can be applied to any economy with welfare-relevant pecuniary externalities to provide simple and intuitive formulas for optimal policy intervention. Examples include externalities arising from wage changes, terms of trade fluctuations, or exchange rate movements. The general principle is that a planner wants to tax actions that redistribute wealth away from imperfectly insured agents or that tighten binding financial constraints in proportion to how much the action improves insurance or smoothing.

However, even when pecuniary externalities are present, our paper shows that determining their sign is not straightforward. Two of the three sufficient statistics that determine the sign of distributive pecuniary externalities can flip sign in response to changes in fundamental parameters, as we carefully illustrate in our applications, making it impossible to sign pecuniary externalities in general. Furthermore, even though there is a close relationship between fire sales, financial amplification and the distributive and collateral effects that underlie pecuniary externalities, we show that both fire sales and amplification are neither necessary nor sufficient to obtain inefficient pecuniary externalities.

Our results also provide direct guidance to financial regulators who work on designing so-called “macroprudential” financial regulations with the goal of reducing fire sales and financial amplification to enhance financial stability. Our paper shows that fire sales are only constrained inefficient if they occur between agents who are imperfectly insured or if financial constraints depend on the prices of fire-sale assets. Regulators should thus pay attention to improving insurance of financially constrained agents (e.g. by promoting contingent forms of financing) and stabilizing the value of assets used as collateral (e.g. by adjusting margins in response to asset price movements). This also suggests that it is dangerous if policymakers impose regulatory constraints that explicitly depend on market prices. We hope that our findings help to discipline the ongoing debate on the design of our financial architecture and macroprudential regulation.

## APPENDIX

### APPENDIX A. PROOFS AND DERIVATIONS FROM SECTIONS 2 AND 3

*Proof of Lemma 1.* Equation (14) follows from taking the partial derivatives of the value function (11), exploiting agents’ privately optimal decisions, and applying the definitions of  $\mathcal{D}_{N_j}^{i,\omega}$ ,  $\mathcal{D}_{K_j}^{i,\omega}$ ,  $\mathcal{C}_{N_j}^{i,\omega}$ , and  $\mathcal{C}_{K_j}^{i,\omega}$  in equations (16) to (19). Equation (15) uses the definition of net worth from equation (10), which implies that the total derivative of the value function with respect to  $K_1^i$  has two components. The first one captures the marginal increase in net worth that leads to the same redistribution  $\mathcal{D}_{N_j}^{i,\omega}$  effect as any other change in sector-wide net worth. The second one results from the direct effect of  $K_1^i$  on market prices.

*Proof of Proposition 1.* The Lagrangian corresponding to problem (24) can be written as

$$\begin{aligned} \mathcal{L} = & \sum_i \theta^i \left\{ u^i \left( C_0^i \right) + \eta_0^i C_0^i + \beta \mathbb{E}_0 \left[ V^{i,\omega} \left( N^{i,\omega}, K_1^i, N^\omega, K_1 \right) \right] + \kappa_1^i \Phi_1^i \left( X_1^i, K_1^i \right) \right\} \\ & - v_0 \sum_i \left[ C_0^i + h^i \left( K_1^i \right) - c_0^i \right] - \sum_\omega \nu_1^\omega \sum_i X_1^{i,\omega} \end{aligned}$$

where  $N^{i,\omega} = e_1^{i,\omega} + F_1^{i,\omega}(K_1^i) + X_1^{i,\omega}$  and  $N^\omega = (N^{b,\omega}, N^{\ell,\omega})$ . The set of necessary conditions for the optimality of the constrained planner's problem are

$$\begin{aligned} \frac{d\mathcal{L}}{dC_0^i} &= \theta^i \left[ u^{i'}(C_0^i) + \eta_0^i \right] - v_0 = 0, \quad \forall i \\ \frac{d\mathcal{L}}{dX_1^{i,\omega}} &= -v_1^\omega + \theta^i \beta V_n^{i,\omega} + \theta^i \kappa_1^i \Phi_{1x}^i + \beta \sum_j \theta^j V_{N^j}^{j,\omega} = 0, \quad \forall i, \omega \\ \frac{d\mathcal{L}}{dK_1^i} &= -v_0 h^{i'}(K_1^i) + \theta^i \beta \mathbb{E}_0 \left[ V_n^{i,\omega} F_1^{i,\omega'}(K_1^i) + V_k^{i,\omega} \right] + \theta^i \kappa_1^i \Phi_{1k}^i + \beta \sum_j \theta^j \mathbb{E}_0 \left[ V_{N^j}^{j,\omega} F_1^{j,\omega'}(K_1^i) + V_{K^j}^{j,\omega} \right] = 0, \quad \forall i \end{aligned}$$

a) Using the definition of  $\lambda_i^{i,\omega}$ , the first optimality condition implies  $v_0 = \theta^i \lambda_0^i, \forall i$ . This implies that  $\theta^b / \theta^\ell = \lambda_0^b / \lambda_0^\ell$  as stated in the proposition. Equation (25) follows from dividing the second optimality condition by  $\theta^i$  and using  $\theta^i = v_0 / \lambda_0^i$  from the first optimality condition as well as the envelope condition  $V_n^{i,\omega} = \lambda_1^{i,\omega}$ . Equation (26) follows from substituting  $v_0$  from the first optimality condition into the third optimality condition and using the envelope condition  $V_k^{i,\omega} = \mathbb{E}_0[\lambda_1^{i,\omega} q^\omega]$ .

b) Substituting the tax rates from the proposition into the optimality conditions of private agents with taxes<sup>25</sup>

$$\begin{aligned} (m_1^\omega + \tau_x^{i,\omega}) \lambda_0^i &= \beta \lambda_1^{i,\omega} + \kappa_1^i \Phi_{1x}^i, \quad \forall i, \omega \\ [h^{i'}(K_1^i) + \tau_k^i] \lambda_0^i &= \mathbb{E}_0 \left[ \beta \lambda_1^{i,\omega} \left( F_1^{i,\omega'}(K_1^i) + q^\omega \right) \right] + \kappa_1^i \Phi_{1k}^i, \quad \forall i \end{aligned}$$

replicates the planner's optimality conditions (25) and (26). The lump sum transfers ensure that the budget constraints of private agents are met for the desired allocation. In conjunction with the government budget constraint, this guarantees that the date 0 resource constraint holds. The resulting allocation is thus constrained efficient.

*Proof of Proposition 2.* It follows from equations (29) and (30) by substituting the definition of distributive and collateral effects from equations (16) to (19).

*Proof of Corollary 1.* For collateral externalities, sufficient statistic  $C1$ , corresponding to the shadow value of the financial constraint  $\kappa_2^{b,\omega}$ , is by definition non-negative;  $C2$ , corresponding to the price derivative of the constraint  $\partial \Phi_2^{b,\omega} / \partial q^\omega$ , is by construction non-negative;  $C3$  for the effect of financial net worth on the price of capital  $\partial q^\omega / \partial N^{b,\omega}$  is non-negative under Condition 1. Therefore the product of the three is non-negative under the condition. Sufficient statistic  $C3$  for the effect of sector-wide capital holdings on the price of capital  $\partial q^\omega / \partial K_1^b$  cannot be signed in general – in Application 3 we provide an example of  $\partial q / \partial K_1^b < 0$  whereas in Application 4 we provide an example of  $\partial q / \partial K_1^b > 0$ . The collateral externalities of capital can thus take on either sign.

For distributive externalities, sufficient statistic  $D1$ , corresponding to  $\Delta MRS^{ij,\omega}$ , can take positive or negative values, as we illustrate in Application 3;  $D2$ , corresponding to  $\Delta K_2^{i,\omega}$  and  $X_2^i$ , can take positive or negative values, as we illustrate in Application 2. Even though  $D3$  pins down  $\partial q^\omega / \partial N^{b,\omega}$ , the product of the three sufficient statistics can take on either sign; thus “anything goes” for distributive externalities.

*Proof of Corollary 2.* It follows from no-arbitrage considerations.

*Proof of Corollary 3.* The optimal corrective taxes  $\tau_x^{i,\omega}$  and  $\tau_k^i$  on financing and investment decisions in the general case satisfy

$$\tau_k^i = \mathbb{E}_0 \left[ F_1^{i,\omega'}(\cdot) \tau_x^{i,\omega} \right] + \Xi_i, \quad \forall i \quad (\text{A44})$$

where we define  $\Xi_i$  as the direct effect of the level of capital on collateral and distributive externalities for a given level of net worth

$$\Xi_i := \mathbb{E}_0 \left[ \Delta MRS^{ij,\omega} \left( \frac{\partial q^\omega}{\partial K_1^i} \Delta K_2^{i,\omega} + \frac{\partial m_2^\omega}{\partial K_1^i} X_2^{i,\omega} \right) - \kappa_2^{i,\omega} \frac{\partial \Phi_2^{b,\omega}}{\partial q^\omega} \frac{\partial q^\omega}{\partial K_1^b} \right]$$

Equation (A44) follows by combining equations (18) and (19) with (29) and (30). Equation (32) in the text is a special case of equation (A44) when  $\Xi_i = 0$ .

25. In the particular case of collateral constraints, there exists a relation between the shadow value of a binding collateral constraint and the MRS across dates/states of a given agent, but this is not a robust feature of models with binding price-dependent constraints. Importantly, because collateral effects are not zero-sum on the aggregate, collateral externalities cannot in general be expressed as a *difference* of MRS between agents.

*Proof of Corollary 4.* It follows from Propositions 1 and 2.

*Proof of Corollary 5.* Explicitly substituting  $\lambda_0^i$ , we can write the planner's two main optimality conditions as

$$\left(m_1^\omega + \tau_x^{i,\omega}\right) \left[u^{i'}\left(c_0^i\right) + \eta_0^i\right] = \beta \lambda_1^{i,\omega} + \kappa_1^i \Phi_{1x^\omega}^i \quad \forall i, \omega \quad (\text{A45})$$

$$\left[h^{i'}\left(k_1^i\right) + \tau_k^i\right] \left[u^{i'}\left(c_0^i\right) + \eta_0^i\right] = \mathbb{E}_0 \left[\beta u^{i'}\left(c_1^{b,\omega}\right) \left(F_1^{i,\omega'}\left(k_1^{i,\omega}\right) + q^\omega\right)\right] + \kappa_1^i \Phi_{1k}^i \quad \forall i \quad (\text{A46})$$

For a given real allocations in the economy, these two optimality conditions as well as all the constraints on the planner's problem continue to be satisfied if

- 1 we vary  $m_1^\omega$  and  $\tau_x^{i,\omega}$  in a given state  $\omega$  such that the sum  $(m_1^\omega + \tau_x^{i,\omega})$  remains unchanged for all  $i \in I$ ,
- 2 if  $\eta_0^i > 0$  and we jointly vary  $\tau_x^{i,\omega}$ ,  $\tau_k^i$  and  $\eta_0^i$  such that both  $(m_1^\omega + \tau_x^{i,\omega})[u^{i'}(c_0^i) + \eta_0^i]$ ,  $\forall \omega$  and  $[h^{i'}(k_1^i) + \tau_k^i][u^{i'}(c_0^i) + \eta_0^i]$  remain unchanged for a given agent  $i$ , or
- 3 if  $\kappa_1^b(z) > 0$  for the  $z$ 'th element of the vector  $\kappa_1^b$  and we jointly vary  $\kappa_1^b(z)$  and both  $\tau_k^b$  (if  $\Phi_{1k}^b(z) > 0$ ) and  $\tau_x^{b,\omega}$  (for all  $\omega$  for which  $\Phi_{1x^\omega}^b(z) > 0$ ) such that both  $\kappa_1^b(z) \Phi_{1x^\omega}^b(z) - (m_1^\omega + \tau_x^{b,\omega}) \lambda_0^b$ ,  $\forall \omega$  and  $\kappa_1^b(z) \Phi_{1k}^b(z) - [h^{b'}(k_1^b) + \tau_k^b] \lambda_0^b$  remain unchanged.

Each of the three described variations of implementation consists of changes in tax rates, market prices and shadow prices such that a given optimal real allocation continues to satisfy the planner's optimality conditions, proving the indeterminacy part of the corollary. It follows a fortiori that the planner can employ the three described degrees of freedom to

- 1 set  $\tau_x^{i,\omega} = 0$  for one of the types of agents  $i \in \{b, \ell\}$  in each  $\omega \in \Omega$ ,
- 2 if  $\eta_0^i > 0$ , set either  $\tau_k^i = \hat{\tau}$  (as long as the shadow price  $\eta_0^i$  remains non-negative, i.e. for any  $\hat{\tau}$  such that  $[\hat{\tau} - \tau_k^i] u^{i'}(c_0^i) \leq [h^{i'}(k_1^i) + \tau_k^i] \eta_0^i$  at the original implementation) or  $\tau_x^{i,\omega} = \hat{\tau}$  in one  $\omega \in \Omega$  (for any  $\hat{\tau}$  that satisfies  $[\hat{\tau} - \tau_x^{i,\omega}] u^{i'}(c_0^i) \leq \tau_x^{i,\omega} [m_1^\omega + \tau_x^{i,\omega}] \eta_0^i$  at the original implementation),
- 3 if  $\kappa_1^b(z) > 0$ , set either  $\tau_k^b = \hat{\tau}$  if  $\Phi_{1k}^b(z) > 0$  (for any  $\hat{\tau}$  such that  $(\tau_k^b - \hat{\tau}) \lambda_0^b / \Phi_{1k}^b(z) \leq \kappa_1^b(z)$  at the original allocation) or set  $\tau_x^{b,\omega} = \hat{\tau}$  in any state  $\omega$  for which  $\Phi_{1x^\omega}^b(z) > 0$  (for any  $\hat{\tau}$  that satisfies  $(\tau_x^{b,\omega} - \hat{\tau}) \lambda_0^b / \Phi_{1x^\omega}^b(z) \leq \kappa_1^b(z)$  at the original allocation),

while adjusting the remaining policy instruments and prices to satisfy the planner's optimality conditions (A45) and (A46). The conditions in parentheses ensure that the respective binding constraints in points 2. and 3. continue to be binding so shadow prices do not become negative, i.e. that  $c_0^i = 0$  continues to be satisfied in point 2. and that the  $z$ 'th element of the constraint continues to bind, i.e.  $\Phi_{1k}^b(z) = 0$  in point 3. Moreover, in point 3., only those decision variables that are affected by the binding constraint are included in the indeterminacy, i.e. only the tax on financing decisions in those states of nature  $\omega$  for which  $\Phi_{1x^\omega}^b(z) > 0$  is indeterminate, and the tax on investment is only indeterminate if  $\Phi_{1k}^b(z) > 0$ .

A simple example of the third indeterminacy is when the borrowing constraint  $\Phi_1^b(x_1^{b,\omega}, k_1^b) := (x_1^{b,\omega} + \phi) \geq 0$  is strictly binding in a given state  $\omega$ . In that case, a marginal change in the tax rate  $\tau_x^{b,\omega}$  changes the shadow price  $\kappa_1^b$  but does not have any real effects. If the constraint is tight enough, the tax can be set to  $\tau_x^{b,\omega} = 0$  in that state without any effect on the real allocation, and the financing decision for state  $\omega$  can be considered constrained efficient. In this example, the constraint does not depend on capital,  $\Phi_{1k}^b = 0$ , so the optimal tax rate on capital is unchanged when we vary  $\tau_x^{b,\omega}$ . We utilize this example in Application 3.

## APPENDIX B. PROOFS AND DERIVATIONS OF APPLICATIONS IN SECTION 4

### Appendix B.1. Analytic Details on Application 1

*Characterizing uniqueness at date 1.* Assume that borrowers and lenders enter date 1 with state variables  $(n, k_1; N, K_1)$  where  $n = N$  and  $k_1 = K_1$ . Let us denote by  $z$  the amount of resources that borrowers receive from lenders at date 1 – both via borrowing and fire sales – and by  $\rho(z)$  the resulting payoff received by lenders at date 2 – both from the repayment of borrowing and from production using fire-sold assets

$$z = m_2 x_2^\ell + q k_2^\ell \quad (\text{B47})$$

$$\rho(z) = x_2^\ell + F^\ell(k_2^\ell) \quad (\text{B48})$$

Given this notation, we can describe  $\rho(z)$  as the “supply of funds” of lenders. Let us also denote by  $\gamma(z)$  the total resources given up by borrowers at date 2 – both as a repayment and because of production foregone – and by  $\delta(z)$  the deadweight loss of fire sales that results from the lower productivity of lenders

$$\begin{aligned}\gamma(z) &= x_2^\ell + A_2 k_2^\ell \\ \delta(z) &= \gamma(z) - \rho(z) = A_2 k_2^\ell - F^\ell(k_2^\ell)\end{aligned}$$

The market prices  $m_2$  and  $q$  are pinned down by the optimality conditions of lenders

$$m_2 = \frac{u'(e_2^\ell + \rho(z))}{u'(n^\ell - z)} \quad (\text{B49})$$

$$q = m_2 F^{\ell'}(k_2^\ell) \quad (\text{B50})$$

Our goal is to formally describe the conditions under which the “supply of funds” curve of lenders is well-behaved so as to lead to a unique equilibrium. Given our assumptions on production technology, there are two distinct regions for the date 1 equilibrium.

*Unconstrained equilibrium.* When the financial constraint is slack, then equations (B47) to (B50) together with  $k_2^\ell = 0$  (or, equivalently,  $z = m_2 x_2^\ell$ ) define a system of 5 equations in 6 variables  $(z, \rho, m_2, q, x_2^\ell, k_2^\ell)$ . We reduce the system to a single implicit equation,

$$zu'(n^\ell - z) = \rho u'(e_2^\ell + \rho)$$

which defines a supply of funds curve  $\rho = \rho(z)$  by lenders that satisfies

$$\frac{\partial \rho}{\partial z} = \frac{u'(c_1^\ell) - zu''(c_1^\ell)}{u'(c_2^\ell) + \rho u''(c_2^\ell)} > 0$$

and is non-degenerate as long as

$$\eta_{c_2} \frac{\rho}{c_2} < 1 \quad (\text{B51})$$

where  $\eta_{c_2} := -c_2^\ell u''(c_2^\ell) / u'(c_2^\ell)$ . If condition (B51) is satisfied, there exists a unique equilibrium.

*Constrained equilibrium.* When the constraint is binding, then equations (B47) to (B50) together with  $x_2^\ell = m_2 \phi A_2 (k_1^b - k_2^\ell)$  defines a system of 5 equations in 6 variables  $(z, \rho, m_2, q, x_2^\ell, k_2^\ell)$ . Combining them, we find

$$zu'(n^\ell - z) = u'(e_2^\ell + \phi A_2 (k_1^b - k_2^\ell) + F^\ell(k_2^\ell)) [\phi A_2 (k_1^b - k_2^\ell) + k_2^\ell F^{\ell'}(k_2^\ell)]$$

This equation implicitly defines a “demand for fire sales” curve  $k_2^\ell = k(z)$  that satisfies

$$\frac{\partial k}{\partial z} = \frac{u'(c_1^\ell) - zu''(c_1^\ell)}{u'(c_2^\ell) [F^{\ell'}(k_2^\ell) - \phi A_2 + k_2^\ell F^{\ell''}(k_2^\ell)] + u''(c_2^\ell) [F^{\ell'}(k_2^\ell) - \phi A_2] \rho} > 0$$

and is non-degenerate as long as the denominator is positive, which requires two conditions,

$$\eta_{qk_2} + \frac{\phi A_2}{F^{\ell'}(k_2^\ell)} < 1 \quad (\text{B52})$$

$$\eta_{c_2} \frac{\rho}{c_2} \left[ 1 - \frac{\eta_{qk_2}}{1 - \frac{\phi A_2}{F^{\ell'}(k_2^\ell)} - \eta_{qk_2}} \right] < 1 \quad (\text{B53})$$

where  $\eta_{qk_2} := -k_2^\ell F^{\ell''}(k_2^\ell) / F^{\ell'}(k_2^\ell)$  so that the first square bracket is positive, and where the term in square brackets derives from  $\frac{F^{\ell'}(k_2^\ell) - \phi A_2}{F^{\ell'}(k_2^\ell) - \phi A_2 + k_2^\ell F^{\ell''}(k_2^\ell)} = 1 - \frac{k_2^\ell F^{\ell''}(k_2^\ell)}{F^{\ell'}(k_2^\ell)} / \left( 1 - \frac{\phi A_2}{F^{\ell'}(k_2^\ell)} - \eta_{qk_2} \right)$ .

Under the three conditions (B51), (B52), and (B53), the function  $k(z)$  is well-behaved and captures how much capital borrowers have to give up to raise  $z$  units of funds at date 1 when the constraint is binding. The function is defined up to an upper limit  $z^{\max}$  that is given by  $k(z^{\max}) = k_1^b$ .

It is now straightforward to express the supply of funds curve  $\rho(z)$  of lenders as well as the deadweight loss curve  $\delta(z)$  and the resources given up by borrowers  $\gamma(z)$  by substituting  $x_2^\ell = \phi A_2 (k_1^b - k_2^\ell)$  and  $k_2^\ell = k(z)$  into the three expressions to obtain

$$\begin{aligned}\rho(z) &= x_2^\ell + F^\ell(k_2^\ell) = \phi A_2 (k_1^b - k(z)) + F^\ell(k(z)) \\ \delta(z) &= A_2 k_2^\ell - F^\ell(k_2^\ell) = A_2 k(z) - F^\ell(k(z)) \\ \gamma(z) &= x_2^\ell + A_2 k_2^\ell = \phi A_2 k_1^b + (1 - \phi) A_2 k(z)\end{aligned}$$

Given that  $k(z)$  is non-degenerate and strictly increasing, all three functions are well-defined and strictly increasing. Note that condition (B52) implies that  $F^{\ell'}(k_2^\ell) > \phi A_2$ .

In combination, under the stated assumptions the two regions, constrained and unconstrained, define a supply of funds curve  $\rho(z)$  of lenders that is strictly increasing and continuous over the interval  $z \in [0, z^{\max}]$ , which is sufficient to guarantee existence and equilibrium uniqueness at date 1. In conjunction with equations (B49) and (B50), this also ensures that Condition 1 is satisfied.

*Date 0.* Given the lack of financial frictions, it is trivial to close the model at date 0 after defining conditions for uniqueness of the date 1 equilibrium. Hence, for brevity, we omit the details of the date 0 characterization.

## Appendix B.2. Analytic Details on Application 2

*Date 1 lenders' problem.* The date 1 value function of lenders is

$$V^{\ell,\omega}(n^{\ell,\omega}, k_1^\ell; N^\omega, K_1) = \max_{k_2^{\ell,\omega}} n^{\ell,\omega} + q^\omega (k_1^\ell - k_2^{\ell,\omega}) + F_2^\ell(k_2^{\ell,\omega})$$

Their optimality condition is given by equation (34). The partial derivatives of  $V^{\ell,\omega}$  internalized by private agents are given by

$$\begin{aligned}V_n^{\ell,\omega} &= \lambda_1^\ell = 1 \\ V_k^{\ell,\omega} &= q^\omega\end{aligned}$$

*Date 1 borrowers' problem and equilibrium.* The value function of borrowers is given by equation (36). If  $N^{b,\omega} \geq \hat{N}$ , then equilibrium is unconstrained so  $q^\omega = A_2$  and the value function of borrowers can be simplified to

$$V^{b,\omega}(n^{b,\omega}, k_1^b; N^\omega, K_1) = n^{b,\omega} + A_2 k_1^b$$

where the partial derivatives of  $V^{b,\omega}$  internalized by private agents are given by  $V_n^{b,\omega} = \lambda_1^{b,\omega} = 1$ ,  $V_k^{b,\omega} = A_2$ , and  $V_{N^j}^{b,\omega} = V_{K^j}^{b,\omega} = 0$ .

If  $N^{b,\omega} \leq \hat{N}$ , then equilibrium is constrained, and the value function of borrowers is

$$V^{b,\omega}(n^{b,\omega}, k_1^b; N^{b,\omega}, K_1^\ell) = A_2 \left( k_1^b + \frac{n^{b,\omega}}{q(N^{b,\omega}, K_1^\ell)} \right)$$

The partial derivatives of this value function that are internalized by private agents are

$$\begin{aligned}V_n^{b,\omega} &= \frac{A_2}{q(N^{b,\omega}, K_1^\ell)} = \frac{1 + K_1^\ell}{1 + \frac{N^{b,\omega}}{A_2}} \\ V_k^{b,\omega} &= A_2\end{aligned}$$

The uninternalized distributive effects are

$$\begin{aligned}\mathcal{D}_{N^b}^{b,\omega} &= -\frac{\partial q^\omega}{\partial N^{b,\omega}} (K_2^{b,\omega} - K_1^b) = -\frac{1}{1 + K_1^\ell} \frac{N^{b,\omega} (1 + K_1^\ell)}{N^{b,\omega} + A_2} = -\frac{N^{b,\omega}}{N^{b,\omega} + A_2} \\ \mathcal{D}_{K_1^\ell}^{b,\omega} &= -\frac{\partial q^\omega}{\partial K_1^\ell} (K_2^{b,\omega} - K_1^b) = \frac{N^{b,\omega} + A_2}{(1 + K_1^\ell)^2} \frac{N^{b,\omega} (1 + K_1^\ell)}{N^{b,\omega} + A_2} = \frac{N^{b,\omega}}{1 + K_1^\ell}\end{aligned}$$

It can easily be verified that  $V_{N^b}^{b,\omega} = \lambda_1^b \mathcal{D}_{N^b}^{b,\omega}$  and  $V_{K_1^\ell}^{b,\omega} = \lambda_1^b \mathcal{D}_{K_1^\ell}^{b,\omega}$ . As described in the text, whenever  $N^{b,\omega} \geq N^{b,\min}$ , there exists a unique equilibrium at date 1.

*Date 0 equilibrium.* For both sets of agents, optimal date 0 capital investment is determined by

$$\max_{k_1^i} \mathbb{E} \left[ V^{i,\omega} \left( A_1^\omega k_1^i - h \left( k_1^i \right), k_1^i; N^b, K_1^\ell \right) \right]$$

with optimality condition

$$\mathbb{E} \left[ \lambda_1^{i,\omega} \left[ A_1^\omega - h' \left( k_1^i \right) \right] + V_k^{i,\omega} \right] = 0$$

Under the assumption that the financial constraint is slack in the high state and binding in the low state. For lenders, the optimality condition is then

$$h' \left( K_1^\ell \right) = \alpha K_1^\ell = \mathbb{E} \left[ A_1^\omega + q^\omega \right] = \mathbb{E} \left[ A_1^\omega \right] + (1 - \pi) A_2 + \pi q^L \left( N^{b,L}, K_1^\ell \right) \quad (\text{B54})$$

The left-hand side of this expression is increasing in  $K_1^\ell$  and the right-hand side is decreasing in  $K_1^\ell$  (since  $\partial q / \partial K_1^\ell < 0$ ), pinning down a unique solution for  $K_1^\ell$ . For borrowers, the date 0 optimality condition is

$$(1 - \pi) \left[ A_1^H - \alpha K_1^b + A_2 \right] = \pi \left\{ \frac{A_2}{q^L \left( N^{b,L}, K_1^\ell \right)} \left[ \alpha K_1^b - A_1^L \right] - A_2 \right\} \quad (\text{B55})$$

The left-hand side captures the marginal gains from additional capital investment in the high state of nature, which, in equilibrium, must be positive and must offset the marginal losses from additional investment in the low state of nature, captured by the right-hand side. The optimum thus needs to satisfy  $\alpha K_1^b \in [A_1^L, A_1^H + A_2]$ . Since the left-hand side is decreasing in  $K_1^b$  and the right-hand side is increasing in  $K_1^b$ , the optimality condition pins down a unique solution within this interval.

The condition under which the constraint is indeed slack in a given state is  $N^{b,\omega} = A_1^\omega K_1^b - \alpha \left( K_1^b \right)^2 / 2 \geq A_2 K_1^\ell = \hat{N}^b$  or

$$A_1^\omega \geq \frac{A_2 K_1^\ell}{K_1^b} + \frac{\alpha K_1^b}{2}$$

Intuitively, the return on capital in the high state needs to cover both the additional capital purchases from lenders (per unit of borrower capital) and the average cost of investment. We assume that this inequality is satisfied in the high state but violated in the low state of nature.<sup>26</sup>

*Proof of Application 2.* Given all other parameters, we define the threshold  $\tilde{A}_1^L$  such that  $N^{b,L} = 0$  or, equivalently,

$$\tilde{A}_1^L = \frac{\alpha K_1^b}{2}$$

This condition together with the optimality conditions (B54) and (B55) pins down a unique level of  $\tilde{A}_1^L$ . By construction,  $N^{b,L} = 0$  for  $A_1^L = \tilde{A}_1^L$ , proving case 2 of the proposition. Standard stability conditions imply that  $dN^{b,L} / dA_1^L > 0$ . As a result,  $A_1^L < \tilde{A}_1^L$  implies  $N^{b,L} < \hat{N}^b$  and  $A_1^L > \tilde{A}_1^L$  leads to  $N^{b,L} > \hat{N}^b$ , proving the other two cases. In the limit case  $\pi \rightarrow 0$ , the threshold is easy to characterize,  $\tilde{A}_1^L = \alpha K_1^b / 2 = (A_1^H + A_2) / 2$ .

### Appendix B.3. Analytic Details on Application 3

*Date 1 lenders' problem.* The date 1 value function of lenders is

$$V^\ell \left( n^\ell, k_1^\ell; N, K_1 \right) = \max_{k_2^\ell} n^\ell + q \left( k_1^\ell - k_2^\ell \right) + F_2^\ell \left( k_2^\ell \right)$$

Their optimality condition is given by equation (37). The partial derivatives of  $V^\ell$  internalized by private agents are given by

$$V_n^\ell = \lambda_1^\ell = 1 \quad \text{and} \quad V_k^\ell = q$$

26. In the limit case  $\pi \rightarrow 0$ , optimal investment implies  $\alpha K_1^i = A_1^H + A_2$  for both agents, and the condition simplifies to  $A_1^\omega - (A_1^H + A_2) / 2 \geq A_2$  or

$$A_1^H \geq A_2 \quad \text{and} \quad A_1^L < \frac{A_1^H + 3A_2}{2}$$

*Date 1 borrowers' problem and equilibrium.* The definition of date 1 borrowers' net worth  $n^b = x_1^b$  together with their date 1 budget constraint and financial constraint implies  $c_1^b = n^b + q(k_1^b - k_2^b)$ . The value function of borrowers is thus given by equation (38) and the partial derivatives of this value function that are internalized by private agents are

$$V_n^b = u'(c_1^b) \quad \text{and} \quad V_k^b = qu'(c_1^b)$$

The optimal capital holdings of borrowers are given by

$$q = \frac{u'(c_2^b)}{u'(c_1^b)} A = \frac{n^b + q(k_1^b - k_2^b)}{Ak_2^b} A = \frac{n^b}{2k_2^b - k_1^b}$$

which corresponds to equation (39). Combining lenders' demand (37) and borrowers' supply (39) for capital implies

$$K_2^b = \frac{A + N^b}{2A + N^b} K_1^b \quad \text{and} \quad q = \frac{2A + N^b}{K_1^b}$$

*Date 0 equilibrium.* At date 0, borrowers solve

$$\max_{x_1^b, k_1^b} u(e_0^b - h^b(k_1^b) - x_1^b) + V^b(x_1^b, k_1^b; K, N) + \underline{\lambda}(x_1^b - \phi) + \bar{\lambda}(\bar{\phi} - x_1^b)$$

Their optimality conditions are given by

$$u'(c_0^b) = u'(c_1^b) + \underline{\lambda} - \bar{\lambda}$$

$$h^{b'}(k_1^b) u'(c_0^b) = qu'(c_1^b)$$

Substituting for the date 0 and 1 budget constraints, the second condition can be re-written as  $h^{b'}(K_1^b)u'(e_0^b - h(K_1^b) - X_1^b) = qu'(X_1^b + A)$  and, using the equilibrium  $q$  from above, solved for

$$K_1^b = \sqrt{\frac{(2A + X_1^b)(e_0^b - X_1^b)}{\alpha(1.5X_1^b + 2A)}}$$

If the date 0 financial constraints are slack, we find  $\underline{\lambda} = \bar{\lambda} = 0$ . The Euler equation then implies  $c_0^b = c_1^b$ , and the expression for capital investment simplifies to  $K_1^{b*} = \sqrt{(2A + X_1^b)/\alpha}$ . We can solve for the optimal unconstrained level of saving

$$X_1^{b*} = \frac{e_0^b - 2A}{2.5}$$

The two constraints on borrowing and saving are indeed slack if  $X_1^{b*} \in [\phi, \bar{\phi}]$ . Otherwise, if the constraint on borrowing (saving) is binding, then  $X_1^b = \phi$  (or  $X_1^b = \bar{\phi}$ ). The threshold values  $\underline{e}$  and  $\bar{e}$  for the initial endowment  $e_0^b$  at which the two constraints become binding are defined by  $X_1^{b*} = \phi$  and  $X_1^{b*} = \bar{\phi}$ , respectively.

*Welfare analysis.* The sensitivities of the equilibrium price of capital  $q = (2A + N^b)/K_1^b$  to  $(K_1^b, N^b)$  are given by

$$\frac{\partial q}{\partial K_1^b} = -\frac{2A + N^b}{(K_1^b)^2} < 0$$

$$\frac{\partial q}{\partial N^b} = \frac{1}{K_1^b} > 0$$

and the respective distributive effects are  $\mathcal{D}_{N^b}^{b,\omega} = -\partial q^\omega / \partial N^{b,\omega} \cdot \Delta K_2^{b,\omega} = -\Delta K_2^{b,\omega} / K_1^b > 0$  since  $\Delta K_2^{b,\omega} < 0$  always holds. Note that  $MRS^\ell = 1$  and  $MRS^b = u'(c_1^b)/u'(c_0^b) = 1 - (\underline{\lambda} - \bar{\lambda})/u'(c_0^b)$ . Consequently,

$$\Delta MRS^{b\ell} = \frac{\bar{\lambda} - \underline{\lambda}}{u'(c_0^b)}$$

Therefore, if borrowers are borrowing-constrained at date 0, then  $\underline{\lambda} > 0$  and  $\bar{\lambda} = 0$ , which implies that  $\Delta MRS^{b\ell} < 0$  and  $\tau_k^b < 0$ , so there is under-investment in that case. Instead, if borrowers are saving-constrained at date 0, then  $\underline{\lambda} = 0$  and  $\bar{\lambda} > 0$ , which implies that  $\Delta MRS^{b\ell} > 0$  and  $\tau_k^b > 0$ , so there is over-investment in that case.

#### Appendix B.4. Analytic Details on Application 4

Because lenders are risk neutral and have no use for capital, they simply pin down the equilibrium value of  $m_2 = 1$ . We thus focus exclusively on the borrowers' problem.

*Date 1 borrowers' problem and equilibrium.* The date 1 value function is given by

$$V^b(n^b, k_1^b; N, K_1) = \max_{x_2^b, k_2^b} u(n^b - q\Delta k_2^b - x_2^b) + x_2^b + A_2 k_2^b + \kappa_2^b (x_2^b + \phi q k_2^b)$$

with optimality conditions

$$\begin{aligned} q(u'(c_1^b) - \phi \kappa_2^b) &= A_2 \\ u'(c_1^b) &= 1 + \kappa_2^b \end{aligned}$$

which, combined, yield equation (41) in the text and define a function  $q(C_1^b)$  with  $q'(C_1^b) > 0$ .

In an unconstrained equilibrium, borrowers consume  $C_1^b = 1$  and save  $X_2^b = N^b - 1$  at date 1, resulting in a price of capital  $q = F_2^b(\cdot) = A_2$ . This allocation is first-best and is feasible as long as  $X_2^b \geq -\phi q K_2^b$  or, equivalently,  $N^b \geq 1 - \phi A_2 K_1^b$ .

Otherwise, if  $N^b \in (0, 1 - \phi A_2 K_1^b)$ , then borrowing is constrained to  $X_2^b = -\phi q K_1^b$ . This is the case on which we focus in the main text. The date 1 budget constraint then implies  $C_1^b = N^b + \phi q K_1^b$  which leads to equation (42) in the text and implicitly defines a function  $C_1^b(N^b, K_1^b)$  that satisfies  $\partial C_1^b / \partial N^b, \partial C_1^b / \partial K_1^b > 0$ . In combination with equation (41), this pins down the price of capital  $q(N^b, K_1^b)$  as a function that is strictly increasing in both arguments.

The date 0 optimality conditions of individual agents are given by the standard Euler equation  $u'(C_0^b) = u'(C_1^b)$  or equivalently  $C_0^b = C_1^b$  and the optimality condition for capital investment (43). Combining the condition  $N^b = F(K_1^b) + X_1^b \geq 1 - \phi A_2 K_1^b$  for a slack financial constraint at date 1 with the date 0 budget constraint, we observe that the unconstrained (and first-best) allocation is feasible if  $e_0^b \geq 2 + [(\frac{1}{2} - \phi)A_2 - \frac{1}{2}A_1]K_1^{b*}$ .

In our application, we assume that this inequality is violated so that the financial constraint is binding. In that case, the date 0 and 1 budget constraint, the binding financial constraint, the Euler equation  $C_0^b = C_1^b = C^b$  and the condition for optimal capital investment (43) can be combined to obtain the equilibrium condition

$$2C^b + \left[ q(C^b) \left( \frac{1}{2} - \phi \right) - \frac{A_1}{2} \right] \frac{A_1 + q(C^b)}{\alpha} - e = 0$$

Holding all other parameters constant, this equation defines a continuous equilibrium consumption function  $C^b(A_1)$  that satisfies  $C^b(A_1) > 0$  under the restrictions that we have imposed on  $A_1$  and  $\phi$ . By implication, it gives rise to a strictly increasing asset price function  $q(A_1) = q(C^b(A_1))$  and capital investment  $K_1(A_1) = (A_1 + q(A_1)) / \alpha$  and to a unique threshold  $\hat{A}_1$  at which the  $A_1 + \phi q(A_1) = 0$ .

#### Appendix B.5. Distributive Externalities and Multiple Equilibria

Applications 5 and 6 illustrate that violations of Condition 1 are typically associated with backward-bending demand curves that lead to multiple and locally unstable equilibria.

*Environment.* We modify Application 3 to introduce multiple equilibria. We now assume that borrowers have CRRA utility  $U^b = \sum_{t=0}^2 u(c_t^b)$ , where  $u(\cdot) = c^{1-\theta} / (1-\theta)$ , and that lenders have linear utility  $U^\ell = c_0^\ell + c_1^\ell + c_2^\ell$ , with  $c_t^\ell \geq 0$ . This is a perfect foresight economy with no uncertainty. Lenders have large endowments of the consumption good at each date while borrowers have non-negative endowments  $e_0^b \geq 0, e_1^b = e_2^b = 0$ . Only borrowers invest at date 0. Formally, borrowers' investment technology is given by  $h^b(k) = \alpha k^2 / 2$ , while lenders' technology corresponds to  $h^\ell(k) = \infty$ , for  $k > 0$ . Both borrowers and lenders are unproductive at date 1, but they produce according to  $F_2^b(k) = Ak$  and  $F_2^\ell(k) = A(k + \delta)^\eta / \eta$  at date 2, where  $\eta < 1, A > \alpha$  and  $\delta \geq 0$ , for  $k_2^b > \delta$ . Conceptually, this formulation introduces more curvature into the model. Agents face the same financial constraints as in Application 3.

In the first-best, borrowers' date 0 investment corresponds to  $k_1^b = A / \alpha$ . At date 1, borrowers hold  $k_2^\ell = 1 - \delta$  and  $k_2^b = k_1^b - 1 + \delta$ . Borrowers' consumption is equalized across all dates.

*Date 1 equilibrium and multiplicity.* The date 1 demand for capital assets by lenders is given by their optimality condition

$$q = F_2^{\ell'}(K_2^\ell) = A(K_2^\ell + \delta)^{\eta-1} = A(K_1^b - K_2^b + \delta)^{\eta-1} \quad (\text{B56})$$

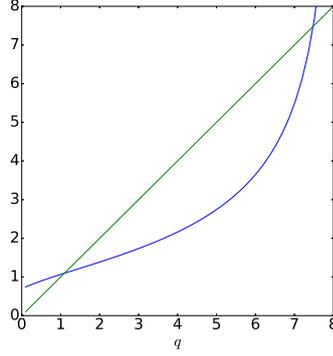


FIGURE 5  
Multiple solutions for (B60)

Given borrowers' net worth, which corresponds to  $n^b = x_1^b$ , their date 1 budget constraint, and financial constraint imply  $c_1^b = n^b + q(k_1^b - k_2^b)$ . Therefore, borrowers' date 1 value function corresponds to

$$V^b(n^b, k_1^b; N, K_1) = \max_{k_2^b} u(n^b + q(k_1^b - k_2^b)) + u(Ak_2^b), \quad (\text{B57})$$

Their optimal capital holdings for borrowers are given by

$$qu'(n^b + q(k_1^b - k_2^b)) = Au'(Ak_2^b) \quad (\text{B58})$$

Setting  $A = 1$  without loss of generality, because of the homogeneity of the problem, we show that the sensitivity of borrowers' demand for capital to prices is determined by

$$\frac{\partial k_2^b}{\partial q} = -\frac{u'(c_1^b) [\theta q (k_1^b - k_2^b) / c_1^b - 1]}{q^2 u''(c_1^b) (k_1^b - k_2^b) + u''(c_2^b)}$$

When income effects are sufficiently strong,  $\partial k_2^b / \partial q$  can take on positive values, implying that lower prices reduce the demand for capital of borrowers. Formally, this occurs when the curvature of the utility function is sufficiently large

$$\theta - 1 > \frac{n^b}{q(k_1^b - k_2^b)}$$

We can solve equation (B58) for  $K_2^b$  to explicitly find an expression for the demand for capital given  $N^b$  and  $K_1^b$

$$K_2^b = \frac{N^b + qK_1^b}{q^{\frac{1}{\theta}} + q} \quad (\text{B59})$$

The equilibrium of the economy is then fully characterized by the solution to equations (B56) and (B59). Combining both equations we can directly characterize  $q(N^b, K_1^b)$  as the solution to the following equation

$$q = \left( K_1^b - \frac{N^b + qK_1^b}{q^{\frac{1}{\theta}} + q} + \delta \right)^{\eta-1} \quad (\text{B60})$$

For given  $N^b$  and  $K_1^b$ , equation (B60) may have multiple solutions, as illustrated in Figure 5, which depicts the left-hand side and right-hand side of the equation for the parameter values reported at the end of Appendix B. It also follows that that  $\partial q / \partial N^b > 0$  in the equilibrium with low  $q$ , which is the one that survives for any value of  $\theta$  whenever an equilibrium exists.<sup>27</sup> The equilibrium with high price features  $\partial q / \partial N^b < 0$ , which violates Condition 1 in the text.

27. In this specific example, because we have assumed that  $\delta < 0$ , there is also a possibility of nonexistence of equilibrium when  $N^b$  is sufficiently high.

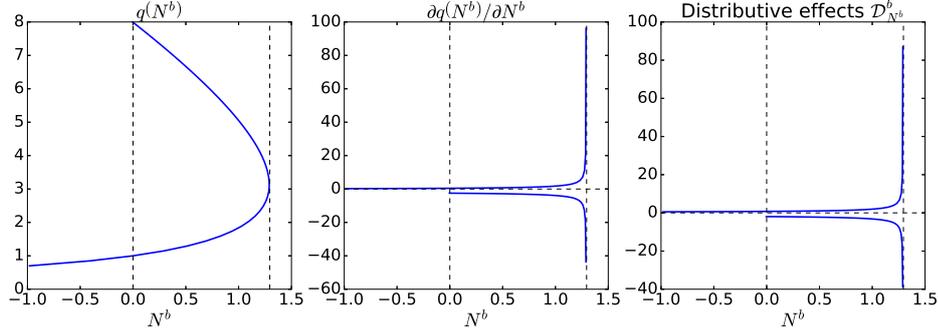


FIGURE 6  
Equilibrium correspondences

Our assumptions guarantee that borrowers are net sellers of capital, so  $\Delta K_2^b < 0$ . The distributive effects of borrowers' net worth, defined in Lemma 1, are given by

$$\mathcal{D}_{N^b}^b = -\frac{\partial q}{\partial N^b} \Delta K_2^b$$

and will therefore have the same sign as  $\frac{\partial q}{\partial N^b}$ . Consequently, for a given sign of  $\Delta MRS^{bl}$ , whose determination is extensively discussed in Application 3, if one equilibrium features overborrowing, the other one will feature underborrowing.

**Application 5. (Changing sign of  $\partial q/\partial N^b$ , distributive externalities).** For sufficiently large values of  $\theta$ , there exist  $N^b$  and  $K_1^b$  such that the economy features multiple equilibria, each of them with different signs of  $\partial q/\partial N^b$ . Therefore, for a given sign of  $\Delta MRS^{bl}$ , if one equilibrium features overborrowing, the other one will feature underborrowing, and vice versa.

For brevity, we do not repeat the date 0 characterization of the equilibrium, which follows the same steps as in Application 3, after accounting for the expectation of equilibrium selection.

Propositions 1 and 2 are valid to characterize the planner's constrained optimum regardless of whether there is a unique equilibrium or multiple equilibria. However, the solution to the constrained planning problem will be generically unique for a given equilibrium selection rule, or if the planner has sufficiently rich policy instruments to implement a specific equilibrium. We leave a more detailed analysis of the implementation of optimal corrective policies with multiple equilibria to future research.

#### Appendix B.6. Collateral Externalities and Multiple Equilibria

*Environment.* We modify Application 4 to introduce multiple equilibria. We show that one of them violates Condition 1 in that the price derivative  $\partial q/\partial N^b$  is positive. We continue to assume a perfect foresight economy in which lenders have large endowments, cannot invest at date 0 so  $h^\ell(k) = \infty$  for  $k > 0$ , have no use for capital  $F_1^\ell(k) := 0$ ,  $\forall t$  and linear utility  $U^\ell = c_0^\ell + c_1^\ell + c_2^\ell$  with  $c_i^\ell \geq 0$ . Given this utility and technology, all distributive effects are zero  $\mathcal{D}_{N^j}^i = \mathcal{D}_{K_1^j}^i = 0$ .

For borrowers, we modify the date 0 and 1 period utility functions to be CRRA instead of log so  $U^b = u(c_0^b) + u(c_1^b) + c_2^b$  with  $u(c) = c^{1-\theta}/(1-\theta)$ . We will focus on the case  $\theta > 1$ , since this increases the curvature of the utility function and naturally prepares the ground for multiplicity in our setting. We assume borrowers have endowments  $e_0^b, e_1^b \in (0, 1)$  and  $e_2^b = 0$ ; they invest at date 0 according to the cost function  $h^b(k) = \alpha k^2/2$  and have linear production function  $F_1^b(k) = Ak$  where we assume  $A \leq \sqrt{\alpha}/2$ . The collateral constraints are the same as in Application 4,  $\Phi_1^b := 0$  and  $\Phi_2^b(x_2^b, k_2^b; q) := x_2^b + \phi q k_2^b \geq 0$  with  $\phi \in (0, 1)$ . The first-best exhibits  $C_0^b = C_1^b = 1$ ,  $K_1^{b*} = 2A/\alpha$  and  $q = A$ .

*Date 1 equilibrium and multiplicity.* The date 1 optimization problem of borrowers is

$$V^b(n^b, k_1^b; N, K_1) = \max_{c_1^b, c_2^b, x_2^b, k_2^b} u(c_1^b) + c_2^b \quad \text{s.t. (3),(4),(40)}$$

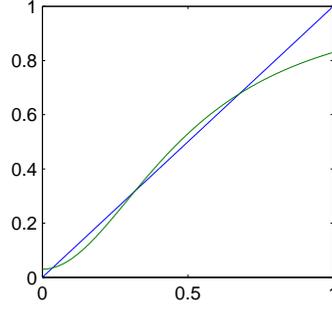


FIGURE 7  
Multiple solutions for (B61)

Since capital always remains in the hands of borrowers, the price of capital as a function of borrower consumption  $C_1^b$  is pinned down by their optimality condition for capital holdings

$$q(C_1^b) = \frac{A}{u'(C_1^b) + \phi\kappa_2^b} = \frac{A}{(1-\phi)(C_1^b)^{-\theta} + \phi}$$

In an unconstrained equilibrium, borrowers consume  $C_1^b = 1$  and save  $X_2^b = N^b - 1$  at date 1, resulting in a price of capital of  $q = F_2^{b'}(\cdot) = A$ . This allocation is feasible as long as  $X_2^b \geq -\phi q K_2^b$  or, equivalently,  $N^b \geq 1 - \phi AK_1^b$ .

In a constrained equilibrium  $X_2^b = -\phi q K_1^b$  and  $C_1^b < 1$ . The date 1 budget constraint then implies that

$$C_1^b = N^b + \phi q(C_1^b) K_1^b = N^b + \frac{\phi AK_1^b}{(1-\phi)(C_1^b)^{-\theta} + \phi} \quad (\text{B61})$$

For given  $N^b$  and  $K_1^b$ , this is an implicit equation in  $C_1^b$  that may have multiple solutions, as illustrated in an example in Figure 7, in which the right-hand side of (B61) is the curved line. Formally, as we vary  $C_1^b$  over the interval  $[0, 1]$ , the collateral term on the right-hand side goes from  $\phi q(0) K_1^b = 0$  (for  $\theta > 1$ ) to  $\phi q(1) K_1^b < 1$ . Multiplicity arises for some values of  $N^b$  if there exists a range of  $C_1^b$  for which the slope of the right-hand side of (B61) exceeds unity, as in the figure. The slope is highest at the inflection point of the price function, i.e. at the value  $\bar{C}$  that satisfies  $q''(\bar{C}) = 0$  and is given by  $\bar{C} = [(\theta - 1)(1 - \phi) / ((\theta + 1)\phi)]^{1/\theta}$ , which we assume w.l.o.g. to be in the unit interval. At that point, the slope of the right-hand side of (B61) is

$$\phi q'(\bar{C}) K_1^b = AK_1^b \left( \frac{\phi}{1-\phi} \right)^{\frac{1}{\theta}} \frac{(\theta+1)^{\frac{\theta+1}{\theta}} (\theta-1)^{\frac{\theta-1}{\theta}}}{4\theta}$$

The first multiplicative term on the right-hand side of this expression is bounded to  $AK_1^b < 1$  by our earlier assumptions; the second and third terms are increasing functions of  $\phi$  and  $\theta$ . If they are chosen sufficiently large, the slope exceeds unity at  $\bar{C}$ , and there is a neighborhood of borrowers' net worth around  $\tilde{N}^b = \bar{C} - \phi q(\bar{C}) K_1^b$  for which equation (B61) has multiple equilibria.

In this area of multiplicity, a given set of state variables  $(N, K_1)$  is consistent with multiple solutions  $C_1^b(N^b, K_1^b)$  and  $q(N^b, K_1^b)$ , as illustrated in Figure 8. The two bottom panels of the figure also show the price derivative  $\partial q / \partial N^b$  with respect to borrowers' net worth and the collateral effects  $C_N^b$ . In the region between the two dashed vertical lines, three equilibria and three possible values for the price derivative and collateral effects exist. As can be seen from the top right panel, the price  $q$  is an increasing function of banker net worth  $N^b$  for two of the three equilibria, but a decreasing function of banker net worth for the middle equilibrium. In the first two equilibria, the standard results on excessive borrowing hold; in the third equilibrium, net worth has negative collateral effects and the decentralized equilibrium exhibits insufficient borrowing.

**Application 6. (Changing sign of  $\partial q / \partial N^b$ , collateral externalities).** *If the parameters  $\phi$  and  $\theta$  are chosen to be sufficiently large to satisfy*

$$AK_1^b \left( \frac{\phi}{1-\phi} \right)^{\frac{1}{\theta}} \frac{(\theta+1)^{\frac{\theta+1}{\theta}} (\theta-1)^{\frac{\theta-1}{\theta}}}{4\theta} > 1$$

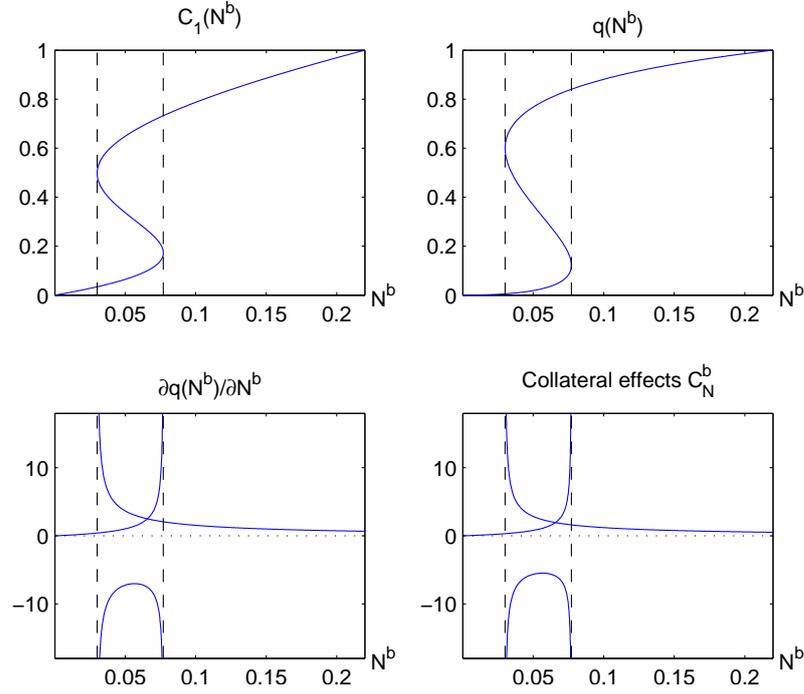


FIGURE 8  
Equilibrium correspondences

then there is a neighborhood of borrowers' net worth around  $\tilde{N}^b$  such that the economy exhibits three equilibria. Two of the three are stable and feature overborrowing and underinvestment; the third equilibrium is unstable and features underborrowing and overinvestment.

In the described setting of multiplicity, the equilibrium with the highest  $C_1^b$  is Pareto-superior to the other two equilibria – the utility of lenders is always constant  $U^\ell = \sum_{t=0}^2 e_t^\ell$  since there are no distributive effects. However, this is not a general feature – when lenders have concave utility or production technologies, distributive effects arise, and lenders may be better off in those equilibria in which borrowers are worse off. A planner can rule out multiple equilibria if she has the capacity to coordinate private agents on one specific equilibrium, for example by committing to a contingent tax/subsidy scheme that makes it suboptimal for individuals to choose inferior equilibria.

For brevity, we do not repeat the date 0 characterization of the equilibrium, which follows the same steps as in Application 4, after accounting for the expectation of equilibrium selection.

#### Parameters Used for Figures in Applications

*Figure 1.* To illustrate Application 1, we assume date 2 production technologies  $F_2^{b,\omega}(k) = k$  and  $F_2^{\ell,\omega}(k) = \log(1+k)$  and period utility functions  $u^i(c) = \log(c)$  with no discounting for both agents. Furthermore we set  $\phi = 0.2$  and date 2 endowments  $e_2^b = 3$  and  $e_2^\ell = 10$ . The figure depicts date 1 equilibrium for  $N^\ell = 2.5$ ,  $K_1^b = 0.5$ ,  $K_1^\ell = 0$  and  $N^b \in [N^{b,\min}, 1.30]$  where the minimum admissible borrower wealth (at which borrowers fire-sell all their capital holdings and obtain non-negative consumption) is  $N^{b,\min} = 0.33$ . The wealth threshold on which the financial constraint on borrowers is marginally binding is  $\tilde{N}^b = 0.8$ .

*Figure 2.* To illustrate Application 2, we set the parameters  $\alpha = 1$ ,  $A_1^H = 3$  and  $A_2 = 1$  and the probability of the low state  $\pi = 5\%$ . We vary  $A_1^L \in [1.2, 3]$  and compute the resulting equilibria, which we trace out in the four panels of the figure. The net trading position of capital  $\Delta K_2^{b,L}$  switches sign at  $\tilde{A}_1^L = 1.8$ .

*Figure 3.* To illustrate Application 3, we set the parameters  $\alpha = \phi = 1/2$ ,  $A = 1$ . We vary  $e_0^b \in [0.25, 2.5]$  and compute the resulting equilibria, tracing out the three sufficient statistics and the resulting tax rate in the four panels of the figure. The borrowing constraint is binding when  $e_0^b < e_0^b = 0.75$  and the constraint on saving is binding when  $e_0^b > e_0^b = 2$ .

*Figure 4.* To illustrate Application 4, we set the parameters  $\alpha = 2$ ,  $A = 1$ ,  $\phi = 1/3$ . We vary  $e_0^b \in [0, 2.5]$  and compute the resulting equilibria, which we trace out in the four panels of the figure. We observe that the collateral constraint is binding when  $e_0^b < e_0^b = 5/3$ .

*Figures 5 and 6.* To illustrate Application 5, we set the parameters  $\eta = 0.4$ ,  $\theta = 2.5$ , and  $\delta = -0.75$ . Figure 5 plots the left- and right-hand-side of equation (B60) for  $N^b = 0.2$  and  $K_1^b = 3.5$ . For Figure 6, we vary  $N^b \in [-1, 1.5]$  and compute all resulting equilibria, which we trace out in the two panels of the figure.

*Figures 7 and 8.* To illustrate Application 6, we set the parameters  $\alpha = 2$ ,  $A = 1$ ,  $\theta = 2$ ,  $\phi = .8$  and  $e_1^b = 0$ . Figure 7 plots the left- and right-hand-side of equation (B61) for  $N^b = 0.03$ . For Figure 8, we vary  $N^b \in [0, 0.22]$  and compute all resulting equilibria, which we trace out in the four panels of the figure.

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